Vrije Universiteit Brussel

Idempotents in group algebras over number fields

joint work with Gabriela Olteanu

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Operations:

$$\left(\sum_{g \in G} r_g g\right) + \left(\sum_{g \in G} s_g g\right) = \sum_{g \in G} (r_g + s_g) g$$

$$\left(\sum_{g \in G} r_g g\right) \cdot \left(\sum_{g \in G} s_g g\right) = \sum_{g,h \in G} r_g s_h gh$$

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Problem

Describe the pci and simple components.

Galois descent

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Let $e(\chi)$ be the pci of $\overline{F}G$ associated to χ and $e_F(\chi)$ the unique pci of FG such that $\chi(e_F(\chi)) \neq 0$.

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Then

$$e(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1})g$$

and

$$e_F(\chi) = \sum_{\sigma \in \mathsf{Gal}(F(\chi)/F)} \sigma \cdot e(\chi).$$

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Disadvantage

No description of simple components!

Previous work

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Our contribution

We extend this to arbitrary number fields F.

Definitions

$\mathcal{C}_F(\mathbb{Z}/k\mathbb{Z})$ = set of orbits of the faithful characters of the cyclic group $\mathbb{Z}/k\mathbb{Z}$ under the action of $\mathrm{Gal}(F(\xi_k)/F)$.

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$\mathcal{C}_F(\mathbb{Z}/k\mathbb{Z}) =$ set of orbits of the faithful characters of the cyclic group $\mathbb{Z}/k\mathbb{Z}$ under the action of $\mathrm{Gal}(F(\xi_k)/F)$.
For $N \trianglelefteq G$ such that $G/N \simeq \mathbb{Z}/k\mathbb{Z}$ and $C \in \mathcal{C}_F(G/N)$, we denote

$$\varepsilon_C(G,N) = \frac{1}{|G|} \sum_{g \in G} \sum_{\psi \in C} \psi(gN) g^{-1}.$$

Abelian groups

Proposition

If G is a finite abelian group and F is a number field, then the map $(N, C) \mapsto \varepsilon_C(G, N)$ is a bijection from the set of pairs (N, C) with $N \trianglelefteq G$, such that G/N is cyclic, and $C \in \mathcal{C}_F(G/N)$ to the set of primitive central idempotents of FG. Further, for every $N \trianglelefteq G$ and $C \in \mathcal{C}_F(G/N)$, $FG\varepsilon_C(G, N) \simeq F(\xi_k)$, where $k = [G : N]$.

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Let e be pci of FG, then $\exists \psi$ linear character of G such that $e = e_F(\psi)$.

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Let e be pci of FG, then $\exists \psi$ linear character of G such that $e = e_F(\psi)$.
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Surjectivity:
Let e be pci of FG, then $\exists \psi$ linear character of G such that $e = e_F(\psi)$. Let $N = \ker(\psi)$ then G/N is cyclic. Define $\chi(gN) = \psi(g)$, a faithful character of G/N with the same character field $F(\chi) = F(\psi)$. Take $C \in \mathcal{C}_F(G/N)$ containing χ, then

$$e = e_F(\psi) = \varepsilon_C(G, N).$$

Some pci for arbitrary groups

Corollary

If G is a finite group with normal subgroup N such that G/N is cyclic of order k and F is a number field, then $\varepsilon_C(G,N)$ is a primitive central idempotent of FG and $FG\varepsilon_C(G,N) \simeq F(\xi_k)$. Furthermore, if $D \in \mathcal{C}_F(G/N)$, then $\varepsilon_C(G,N) = \varepsilon_D(G,N)$ if and only if $C = D$.

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Main observation:
$G \to G/N$ induces a ring isomorphism $\phi : FG\widehat{N} \simeq F(G/N)$ with $\phi(\varepsilon_C(G,N)) = \varepsilon_C(G/N,1)$.

Strong Shoda pairs

If $N \trianglelefteq G$, then

$$\varepsilon(G,N) = \begin{cases} \widetilde{N} & \text{if } N = G \\ \prod_{M/N \in \mathcal{M}(G/N)} (\widetilde{N} - \widetilde{M}) & \text{if } N \neq G \end{cases},$$

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If $K \trianglelefteq H \leq G$, then $e(G,H,K) = \sum_{t \in T} \varepsilon(H,K)^t$ where T is a right transversal of $\mathrm{Cen}_G(\varepsilon(H,K))$ in G.

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A **SSP** of G is a pair (H,K) of subgroups of G satisfying the following conditions:

1. $K \leq H \trianglelefteq N_G(K)$,
2. H/K is cyclic and a maximal abelian subgroup of $N_G(K)/K$,
3. for every $g \in G \setminus N_G(K)$, $\varepsilon(H,K)\varepsilon(H,K)^g = 0$.

More definitions

Let (H, K) be a SSP of G and fix a generator hK of H/K, then $E_F(G, H/K)$

$$= \{g \in N_G(K) \mid g^{-1}hgK = h^iK \text{ for some } i \in \mathrm{Gal}(F(\xi_{[H:K]})/F)\};$$

independent on choice of generator.

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Lemma

If (H, K) is a SSP of G, then $\mathrm{Cen}_G(\varepsilon_C(H, K)) = E_F(G, H/K)$.

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Lemma

If (H, K) is a SSP of G, then $\mathrm{Cen}_G(\varepsilon_C(H, K)) = E_F(G, H/K)$.

In this case, $e_C(G, H, K) = \sum_{t \in T} \varepsilon_C(H, K)^t$ where T is a right transversal of $E_F(G, H/K)$ in G.

Splitting Lemma

1. Let $N \trianglelefteq G$ be such that G/N is cyclic, then

$$\varepsilon(G,N) = \sum_{C \in \mathcal{C}_F(G/N)} \varepsilon_C(G,N).$$

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2. Let $K \leq H \trianglelefteq N_G(K)$ be such that H/K is cyclic and let R be a set of representatives of the action of $N_G(K)$ on $\mathcal{C}_F(H/K)$. Then

$$e(G,H,K) = \sum_{C \in R} e_C(G,H,K).$$

Main Theorem

Let G be a finite group and F be a number field.

1. Let (H, K) be a strong Shoda pair of G and $C \in \mathcal{C}_F(H/K)$. Let $[H : K] = k$ and $E = E_F(G, H/K)$. Then $e_C(G, H, K)$ is a primitive central idempotent of FG and

$$FGe_C(G, H, K) \simeq M_{[G:E]}\left(F\left(\xi_k\right) *_\tau^\sigma E/H\right).$$

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2. Let X be a set of strong Shoda pairs of G. If every primitive central idempotent of $\mathbb{Q}G$ is of the form $e(G, H, K)$ for $(H, K) \in X$, then every primitive central idempotent of FG is of the form $e_C(G, H, K)$ for $(H, K) \in X$ and $C \in \mathcal{C}_F(H/K)$.

Strongly monomial groups

Corollary

If G is a finite strongly monomial group (e.g. abelian-by-supersolvable group / metabelian group) and F a number field. Then every primitive central idempotent if FG is of the form $e_C(G, H, K)$ for a strong Shoda pair (H, K) of G and $C \in \mathcal{C}_F(H/K)$. Furthermore, for every strong Shoda pair (H, K) of G and every $C \in \mathcal{C}_F(H/K)$,

$$FGe_C(G, H, K) \simeq M_{[G:E]}\left(F\left(\xi_{[H:K]}\right) *_\tau^\sigma E/H\right).$$

Number of simple components

Because of the splitting lemma, a pci $e(G, H, K)$ of $\mathbb{Q}G$ splits into m pci of FG where $m =$ number of orbits of the action of $N_G(K)$ on $\mathcal{C}_F(H/K)$.

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$$m = \frac{|\mathcal{C}_F(H/K)|}{|\mathrm{Orb}_{N_G(K)}(C)|} = \frac{\phi([H:K])}{|\mathrm{Gal}(F(\xi_{[H:K]})/F)|} \frac{|E_F(G, H/K)|}{|N_G(K)|}$$

$$= [\mathbb{Q}(\zeta_{[H:K]})^{N_G(K)/H} \cap F : \mathbb{Q}]$$

for any choice of C.

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$$= [\mathbb{Q}(\zeta_{[H:K]})^{N_G(K)/H} \cap F : \mathbb{Q}]$$

for any choice of C.
We say that the number of simple components of FG is **minimal** if $m = 1$ for all pci of $\mathbb{Q}G$.

Sufficient conditions

Corollary

Let G be a finite strongly monomial group and F a number field.

- Let X be a complete set of representatives of strong Shoda pairs of G. If $\phi([H:K]) = |\mathrm{Gal}(F(\xi_{[H:K]})/F)|$ for each $(H,K) \in X$, then the number of simple components of FG is minimal.

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- Let e be the exponent of G. If $\phi(e) = |\mathrm{Gal}(F(\xi_e)/F)|$ then the number of simple components of FG is minimal.

Necessary conditions

In general not necessary, for example for $F = \mathbb{Q}(\zeta_3)$ and $G = D_6$, but it is for abelian groups.

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Corollary

Let G be a finite abelian group of exponent e and F a number field. The number of simple components of FG is minimal if and only if $\phi(e) = |\mathrm{Gal}(F(\xi_e)/F)|$.