Vrije Universiteit Brussel

Finite group rings of nilpotent groups:

a complete set of orthogonal primitive idempotents

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Our aim is to describe a **complete set of orthogonal primitive idempotents** in each simple component of a semisimple group algebra in such a way that it provides a straightforward implementation.

Definition (Olivieri, del Río, Simón; 2004)

A **strong Shoda pair** of G is a pair (H, K) of subgroups of G satisfying

- $K \leq H \trianglelefteq N_G(K)$
- H/K is cyclic and a maximal abelian subgroup of $N_G(K)/K$
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Here, $\varepsilon(H, K)$ is defined as

$$\varepsilon(H,K) = \begin{cases} \widetilde{K} & \text{if } H = K \\ \prod_{M/K \in \mathcal{M}(H/K)} (\widetilde{K} - \widetilde{M}) & \text{if } H \neq K, \end{cases}$$

with $\widetilde{N} = \frac{1}{|N|} \sum_{n \in N} n$ for each group N and $\mathcal{M}(H/K)$ the set of all minimal normal non-trivial subgroups of H/K.

Pci for abelian-by-supersolvable groups

Theorem (Olivieri, del Río, Simón; 2004)

Let G be an **abelian-by-supersolvable** group. Then e is a **pci** of $\mathbb{Q}G$ iff $e = e(G, H, K) := \sum_{t \in T} \varepsilon(H, K)^t$ for a **strong Shoda pair** (H, K) of G.

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Theorem (Olivieri, del Río, Simón; 2004)

Let G be an **abelian-by-supersolvable group** and (H, K) a **strong Shoda pair** of G. Then the **Wedderburn component**

$$\mathbb{Q}Ge(G, H, K) \simeq M_n(\mathbb{Q}(\xi_k) * N_G(K)/H),$$

for $k = [H : K]$ and $n = [G : N_G(K)]$.

Pi for nilpotent groups

Theorem (Jespers, Olteanu, del Río; 2011)

Let G be a **finite nilpotent group**, e a pci of $\mathbb{Q}G$. A **complete set of orthogonal pi** of $\mathbb{Q}Ge$ consists of the conjugates of β by the elements of T, where β and T are **defined algorithmically** in 4 cases.

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Example odd groups

Let G be a finite nilpotent group of **odd order** and $e = e(G, H, K)$ for (H, K) a strong Shoda pair of G, $H/K = \langle \overline{a} \rangle$. Then $\langle \overline{a} \rangle$ has a cyclic complement $\langle \overline{b} \rangle$ in $N_G(K)/K$. A **complete set of orthogonal pi** of $\mathbb{Q}Ge$ consists of **all** G-conjugates of $\widetilde{b}\varepsilon(H, K)$.

q-cyclotomic classes

Let $\mathbb{F}$ be a finite field of order q. Let G be a group of order n, $(n, q) = 1$ and let $\mathbb{Z}_n^*$ be the group of units of $\mathbb{Z}_n$. Define $\mathcal{Q} = \langle \overline{q}_n \rangle \subseteq \mathbb{Z}_n^*$ and consider the action

$$\mathcal{Q} \times G \to G : (\overline{m}, g) \mapsto g^m.$$

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The q-**cyclotomic classes of** G are the orbits of G under the action of $\mathcal{Q}$ on G.

Let $\mathrm{Irr}(G)$ be the group of **irreducible characters** of G. We denote by $\mathcal{C}(G)$ the set of q-cyclotomic classes of $\mathrm{Irr}(G)$ which consist of **linear faithful characters** of G.

Building blocks

Definition (Broche, del Río; 2007)

Let $K \trianglelefteq H \leq G$ be such that H/K is cyclic of order k and $C \in \mathcal{C}(H/K)$. If $\chi \in C$ and $\mathrm{tr} = \mathrm{tr}_{\mathbb{F}(\xi_k)/\mathbb{F}}$ denotes the field trace of the Galois extension $\mathbb{F}(\xi_k)/\mathbb{F}$, then we set

$$\varepsilon_C(H,K) = |H|^{-1} \sum_{h \in H} \mathrm{tr}(\chi(hK))h^{-1}.$$

Furhermore, $e_C(G,H,K)$ denotes the sum of the different G-conjugates of $\varepsilon_C(H,K)$.

Connection $\mathbb{Q}G$ and $\mathbb{F}G$

Theorem (Broche, del Río; 2007)

Let G be a finite group and $\mathbb{F}$ a finite field such that $\mathbb{F}G$ is semisimple. Let X be a set of **strong Shoda pairs** of G. If every **pci of** $\mathbb{Q}G$ is of the form $e(G, H, K)$ for $(H, K) \in X$, then every **pci of** $\mathbb{F}G$ is of the form $e_C(G, H, K)$ for $(H, K) \in X$ and $C \in \mathcal{C}(H/K)$.

Some definitions

Let $K \trianglelefteq H \leq G$ be such that H/K is cyclic. Then $N = N_G(H) \cap N_G(K)$ acts on H/K by conjugation and this induces an **action** of N on the set of q-cyclotomic classes of H/K. Now define $E_G(H, K)$ as the **stabilizer** of any q-cyclotomic class of H/K containing **generators** of H/K.

Pci for abelian-by-supersolvable groups

Theorem (Broche, del Río; 2007)

Let G be an **abelian-by-supersolvable group** and $\mathbb{F}$ a finite field of order q such that $\mathbb{F}G$ is semisimple, then each **pci** of $\mathbb{F}G$ is of the form $e_C(G, H, K)$ for (H, K) a **strong Shoda pair** of G and $C \in \mathcal{C}(H/K)$.
Furthermore, the **Wedderburn component**

$$\mathbb{F}Ge_C(G, H, K) \simeq M_{[G:H]}(\mathbb{F}_{q^{o/[E:K]}}),$$

where $E = E_G(H, K)$ and o the multiplicative order of q modulo $[H : K]$.

Pi for nilpotent groups

Theorem (Olteanu, VG; 2011)

Let $\mathbb{F}$ be a finite field and G a finite **nilpotent group** such that $\mathbb{F}G$ is semisimple and let e be a pci of $\mathbb{F}G$. A **complete set of orthogonal pi** of $\mathbb{F}Ge$ consists of the conjugates of β by the elements of T, where β and T are **defined algorithmically** in 3 cases.

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Structure of the proof

- Follow proof of $\mathbb{Q}G$
- Project pi of $\mathbb{Q}Ge$ on $\mathbb{F}Ge_C$ (Lemma Broche, del Río; 2007)
- Use technical arguments

Problems

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- $G = C_{p,q^m} = \langle a, b \mid a^{q^m} = 1 = b^p, b^{-1}ab = a^r \rangle$, p, q prime: Making use of
  - description of simple components as classical crossed products with trivial twisting
  - explicit isomorphism ψ of classical crossed products with trivial twisting as matrix algebra
  - observe that $\psi(b)$ is permutation matrix and diagonalize $\psi(\widetilde{b}) = PE_{11}P^{-1}$

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  Problem: Describe $\psi^{-1}(P)$ in terms of elements of $\mathbb{Q}G$
- $G = C_{p^2,q} = \langle a, b \mid a^q = 1 = b^{p^2}, b^{-1}ab = a^r \rangle$, p, q prime: Problem: Simple components do not always have trivial twisting

  $\rightarrow$ Needs new idea