Finite group rings of nilpotent groups: a complete set of orthogonal primitive idempotents

Inneke Van Gelder

Groups, Rings, and Group-Rings
University of Alberta, Edmonton, Canada

July 11-15, 2011
Our aim is to describe a complete set of orthogonal primitive idempotents in each simple component of a semisimple group algebra in such a way that it provides a straightforward implementation.
Definition (Olivieri, del Río, Simón; 2004)

A strong Shoda pair of $G$ is a pair $(H, K)$ of subgroups of $G$ satisfying

- $K \leq H \subseteq N_G(K)$
- $H/K$ is cyclic and a maximal abelian subgroup of $N_G(K)/K$
- for every $g \in G \setminus N_G(K)$, $\varepsilon(H, K)\varepsilon(H, K)^g = 0$
Definition (Olivieri, del Río, Simón; 2004)

A strong Shoda pair of $G$ is a pair $(H, K)$ of subgroups of $G$ satisfying

- $K \leq H \leq N_G(K)$
- $H/K$ is cyclic and a maximal abelian subgroup of $N_G(K)/K$
- for every $g \in G \setminus N_G(K)$, $\varepsilon(H, K)\varepsilon(H, K)^g = 0$

Here, $\varepsilon(H, K)$ is defined as

$$\varepsilon(H, K) = \begin{cases} \tilde{K} & \text{if } H = K \\ \prod_{M/K \in \mathcal{M}(H/K)} (\tilde{K} - \tilde{M}) & \text{if } H \neq K, \end{cases}$$

with $\tilde{N} = \frac{1}{|N|} \sum_{n \in N} n$ for each group $N$ and $\mathcal{M}(H/K)$ the set of all minimal normal non-trivial subgroups of $H/K$. 
Pci for abelian-by-supersolvable groups

Theorem (Olivieri, del Río, Simón; 2004)

Let $G$ be an abelian-by-supersolvable group. Then $e$ is a pci of $\mathbb{Q}G$ iff $e = e(G, H, K) := \sum_{t \in T} \varepsilon(H, K)^t$ for a strong Shoda pair $(H, K)$ of $G$. 

Pci for abelian-by-supersolvable groups

Theorem (Olivieri, del Río, Simón; 2004)

Let $G$ be an abelian-by-supersolvable group. Then $e$ is a pci of $\mathbb{Q}G$ iff $e = e(G, H, K) := \sum_{t \in T} \varepsilon(H, K)^t$ for a strong Shoda pair $(H, K)$ of $G$.

Theorem (Olivieri, del Río, Simón; 2004)

Let $G$ be an abelian-by-supersolvable group and $(H, K)$ a strong Shoda pair of $G$. Then the Wedderburn component

$$
\mathbb{Q}Ge(G, H, K) \simeq M_n(\mathbb{Q}(\xi_k) \ast N_G(K)/H),
$$

for $k = [H : K]$ and $n = [G : N_G(K)]$. 
Theorem (Jespers, Olteanu, del Río; 2011)

Let $G$ be a finite nilpotent group, $e$ a pci of $\mathbb{Q}G$. A complete set of orthogonal pi of $\mathbb{Q}Ge$ consists of the conjugates of $\beta$ by the elements of $T$, where $\beta$ and $T$ are defined algorithmically in 4 cases.
Pi for nilpotent groups

Theorem (Jespers, Olteanu, del Río; 2011)
Let $G$ be a finite nilpotent group, $e$ a pci of $\mathbb{Q}G$. A complete set of orthogonal pi of $\mathbb{Q}Ge$ consists of the conjugates of $\beta$ by the elements of $T$, where $\beta$ and $T$ are defined algorithmically in 4 cases.

Example odd groups
Let $G$ be a finite nilpotent group of odd order and $e = e(G, H, K)$ for $(H, K)$ a strong Shoda pair of $G$, $H/K = \langle \bar{a} \rangle$. Then $\langle \bar{a} \rangle$ has a cyclic complement $\langle \bar{b} \rangle$ in $N_G(K)/K$. A complete set of orthogonal pi of $\mathbb{Q}Ge$ consists of all $G$-conjugates of $\tilde{b}\varepsilon(H, K)$. 
Let $\mathbb{F}$ be a finite field of order $q$. Let $G$ be a group of order $n$, $(n, q) = 1$ and let $\mathbb{Z}_n^*$ be the group of units of $\mathbb{Z}_n$. Define $Q = \langle \bar{a}_n \rangle \subseteq \mathbb{Z}_n^*$ and consider the action

\[ Q \times G \rightarrow G : (\bar{m}, g) \mapsto g^m. \]
Let $\mathbb{F}$ be a finite field of order $q$. Let $G$ be a group of order $n$, $(n, q) = 1$ and let $\mathbb{Z}_n^*$ be the group of units of $\mathbb{Z}_n$. Define $Q = \langle \overline{q}_n \rangle \subseteq \mathbb{Z}_n^*$ and consider the action

$$Q \times G \to G : (m, g) \mapsto g^m.$$

**Definition**

The *$q$-cyclotomic classes of* $G$ are the orbits of $G$ under the action of $Q$ on $G$. 
Let $\mathbb{F}$ be a finite field of order $q$. Let $G$ be a group of order $n$, $(n, q) = 1$ and let $\mathbb{Z}_n^*$ be the group of units of $\mathbb{Z}_n$. Define $Q = \langle \overline{q}_n \rangle \subseteq \mathbb{Z}_n^*$ and consider the action $Q \times G \rightarrow G : (\overline{m}, g) \mapsto g^m$.

**Definition**

The *q-cyclotomic classes of* $G$ are the orbits of $G$ under the action of $Q$ on $G$.

Let $\text{Irr}(G)$ be the group of *irreducible characters* of $G$. We denote by $\mathcal{C}(G)$ the set of $q$-cyclotomic classes of $\text{Irr}(G)$ which consist of *linear faithful characters* of $G$. 
Definition (Broche, del Río; 2007)

Let $K \triangleleft H \leq G$ be such that $H/K$ is cyclic of order $k$ and $C \in C(H/K)$. If $\chi \in C$ and $\text{tr} = \text{tr}_{\mathbb{F}(\xi_k)/\mathbb{F}}$ denotes the field trace of the Galois extension $\mathbb{F}(\xi_k)/\mathbb{F}$, then we set

$$\varepsilon_C(H, K) = |H|^{-1} \sum_{h \in H} \text{tr}(\chi(hK))h^{-1}.$$ 

Furthermore, $e_C(G, H, K)$ denotes the sum of the different $G$-conjugates of $\varepsilon_C(H, K)$. 
Connection $\mathbb{Q}G$ and $\mathbb{F}G$

Theorem (Broche, del Río; 2007)

Let $G$ be a finite group and $\mathbb{F}$ a finite field such that $\mathbb{F}G$ is semisimple. Let $X$ be a set of strong Shoda pairs of $G$. If every pci of $\mathbb{Q}G$ is of the form $e(G, H, K)$ for $(H, K) \in X$, then every pci of $\mathbb{F}G$ is of the form $e_C(G, H, K)$ for $(H, K) \in X$ and $C \in C(H/K)$. 
Let $K \trianglelefteq H \leq G$ be such that $H/K$ is cyclic. Then $N = N_G(H) \cap N_G(K)$ acts on $H/K$ by conjugation and this induces an action of $N$ on the set of $q$-cyclotomic classes of $H/K$. Now define $E_G(H, K)$ as the stabilizer of any $q$-cyclotomic class of $H/K$ containing generators of $H/K$. 
Pci for abelian-by-supersolvable groups

Theorem (Broche, del Río; 2007)

Let $G$ be an abelian-by-supersolvable group and $F$ a finite field of order $q$ such that $FG$ is semisimple, then each pci of $FG$ is of the form $e_C(G, H, K)$ for $(H, K)$ a strong Shoda pair of $G$ and $C \in C(H/K)$. Furthermore, the Wedderburn component

$$FG e_C(G, H, K) \simeq M_{[G:H]}(F_{q^o/[E:K]})$$

where $E = E_G(H, K)$ and $o$ the multiplicative order of $q$ modulo $[H : K]$. 
Pi for nilpotent groups

Theorem (Olteanu, VG; 2011)

Let $\mathbb{F}$ be a finite field and $G$ a finite nilpotent group such that $\mathbb{F}G$ is semisimple and let $e$ be a pci of $\mathbb{F}G$. A complete set of orthogonal pi of $\mathbb{F}Ge$ consists of the conjugates of $\beta$ by the elements of $T$, where $\beta$ and $T$ are defined algorithmically in 3 cases.
Pi for nilpotent groups

Theorem (Olteanu, VG; 2011)

Let $\mathbb{F}$ be a finite field and $G$ a finite nilpotent group such that $\mathbb{F}G$ is semisimple and let $e$ be a pci of $\mathbb{F}G$. A complete set of orthogonal pi of $\mathbb{F}Ge$ consists of the conjugates of $\beta$ by the elements of $T$, where $\beta$ and $T$ are defined algorithmically in 3 cases.

Example odd groups

Let $G$ be a finite nilpotent group of odd order and $e = e_C(G, H, K)$ for $(H, K)$ a strong Shoda pair of $G$, $H/K = \langle \bar{a} \rangle$ and $C \in C(H/K)$. Then $\langle \bar{a} \rangle$ has a cyclic complement $\langle \bar{b} \rangle$ in $E_G(H, K)/K$. A complete set of orthogonal pi of $\mathbb{F}Ge$ consists of all $G$-conjugates of $\tilde{b} \in C(H, K)$. 
Structure of the proof

- Follow proof of $\mathbb{Q}G$
- Project $\pi$ of $\mathbb{Q}Ge$ on $\mathbb{F}Ge_C$ (Lemma Broche, del Río; 2007)
- Use technical arguments
Next step: finite metacyclic groups
Problems

Next step: finite metacyclic groups

- $G = C_{p,q^m} = \langle a, b \mid a^{q^m} = 1 = b^p, b^{-1}ab = a^r \rangle$, $p, q$ prime:
  
  Making use of
  
  - description of simple components as classical crossed products with trivial twisting
  - explicit isomorphism $\psi$ of classical crossed products with trivial twisting as matrix algebra
  - observe that $\psi(b)$ is permutation matrix and diagonalize $\psi(\widetilde{b}) = PE_{11}P^{-1}$

Problem: Describe $\psi^{-1}(P)$ in terms of elements of $\mathbb{Q}G$
Next step: finite metacyclic groups

- $G = C_{p,q^m} = \langle a, b \mid a^{q^m} = 1 = b^p, b^{-1}ab = a^r \rangle$, $p, q$ prime:
  - Making use of
    - description of simple components as classical crossed products with trivial twisting
    - explicit isomorphism $\psi$ of classical crossed products with trivial twisting as matrix algebra
    - observe that $\psi(b)$ is permutation matrix and diagonalize $\psi(\tilde{b}) = PE_{11}P^{-1}$

Problem: Describe $\psi^{-1}(P)$ in terms of elements of $\mathbb{Q}G$

- $G = C_{p^2,q} = \langle a, b \mid a^q = 1 = b^{p^2}, b^{-1}ab = a^r \rangle$, $p, q$ prime:
  - Problem: Simple components do not always have trivial twisting
  - Needs new idea