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Universal bialgebras and Hopf algebras associated to Koszul algebras

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Abstract

In this thesis, we study the article 'Representations of non-commutative quantum groups' by Benoit Kriegk and Michel Van den Bergh. In it, they investigate the representation theory (in the comodule sense) of certain universal bialgebras associated to Koszul, and more generally distributive algebras. These bialgebras and the universal property they obey are modeled on a ring of endomorphisms acting on a vector space. The representations are determined by exploiting a connection between distributive algebras and a monoidal category of quiver representations, highly amenable to homological calculations. In the simplest case, that of affine n-space, the associated bialgebra turns out to be a 'bigger' version of the coordinate ring of the monoid of $n \times n$-matrices. We motivate why this bialgebra should be considered as a non-commutative version of this coordinate ring, by linking its representation theory with Young diagrams and Schur-Weyl duality, and by explicitly computing the representation ring using non-commutative symmetric functions. For $n = 2$, we construct the Hopf algebra analog, and show that it can be studied using methods analogous to the ones employed in the representation theory of reductive algebraic groups.

“ . . . I was thinking of the tendency today for people to develop whole areas of mathematics on their own, in a rather abstract fashion. They just go on beavering away. If you ask what is it all for, what is its significance, what does it connect with, you find that they don’t know. . . . Why do we do mathematics? We mainly do mathematics because we *enjoy* doing mathematics. But in a deeper sense, why should we be paid to do mathematics? . . . If mathematics is to be thought of as fragmented specializations, all going off independently and justifying themselves, then it is very hard to argue why people should be paid to do this. We are not entertainers, like tennis players. The only justification is that it is a real contribution to human thought. Even if I’m not directly working in applied mathematics, I feel that I’m contributing to the sort of mathematics that can and will be useful for people who are interested in applying mathematics to other things.”

The Mathematical Intelligencer 6, p. 9-19, 1984

Michael Francis Atiyah

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1

Introduction/Apology

> "How, to have $a \times b$ not equal to $b \times a$ is something that does not fit in very well with geometric ideas; hence his [Einstein] hostility to it."
>
> *The Mathematical Intelligencer 11, p. 58, 1989*
> Paul Adrien Maurice Dirac

The primary goal of this thesis is to elucidate the article 'Representations of non-commutative quantum groups' by Benoit Kriegk and Michel Van den Bergh [51]. Before doing so, let us give some informal geometric motivation. For an algebraically closed field k of characteristic 0, classical algebraic geometry is built upon the following equivalence of categories, and its extension

$$\begin{tikzcd}
\{\text{affine } k\text{-varieties}\} \arrow[r, bend left, "\mathcal{O}(-)"] \arrow[d, hook] & \{\text{reduced affine } k\text{-algebras}\} \arrow[l, bend left, "\mathrm{Spec}(-)"] \arrow[d, hook] \\
\{\text{affine } k\text{-schemes}\} \arrow[r, bend right, "\Gamma(-)"'] & \{k\text{-algebras}\} \arrow[l, bend right]
\end{tikzcd}$$

An alternative approach is based on Grothendieck's functor of points. The intuition is to look at solutions to the polynomial equations defining a k-algebra, not just over the field k itself, but over any (commutative) k-algebra. More formally, to any affine k-scheme $X = \mathrm{Spec}(A)$, one associates the functor

$$\mathrm{Hom}_{k\text{-}\mathsf{commalg}}(A, -) : k\text{-}\mathsf{commalg} \to \mathrm{Sets}$$

Using the Yoneda lemma, one sees that for any other affine k-scheme $Y = \mathrm{Spec}(B)$

$$\mathrm{Nat}(\mathrm{Hom}(A, -), \mathrm{Hom}(B, -)) \cong \mathrm{Hom}(B, A) \cong \mathrm{Mor}(X, Y),$$

so morphisms between functors of points correspond to morphisms between affine k-schemes. In fact, it can be shown that any scheme X (not necessarily affine over k) has an associated functor of points h_X, that

completely characterizes it. One (minor) advantage of this approach is that everything works equally well if A is a non-commutative k-algebra, just by enlarging the category k-$\mathsf{commalg}$ to the category k-alg. Let us now denote by $\mathrm{Spec}(A)$ the functor $\mathrm{Hom}_{k\text{-}\mathsf{alg}}(A, -)$. Some authors use this type of reasoning to justify calling $\mathrm{Spec}(A)$ a 'non-commutative space', though we will generally avoid this terminology.

The Kleinian approach to studying geometric spaces is to look at their symmetry groups or monoids. Consider the simplest spaces: the linear ones. For a k-vector space (say finite dimensional), there is an action

$$\mathrm{End}(V) \times V \to V,$$

and since $\mathrm{End}(V)$ is itself a vector space, and even an algebra, this can be upgraded to a linear map

$$\mathrm{End}(V) \otimes V \to V.$$

It is easy to check that $\mathrm{End}(V)$ is universal among all algebras D equipped with a linear map $D \otimes V \to V$, providing a satisfactory notion of symmetry. Passing over to the coordinate algebras, this induces an algebra map

$$\mathcal{O}(V) \to \mathcal{O}(\mathrm{End}(V)) \otimes \mathcal{O}(V)$$

and $\mathcal{O}(\mathrm{End}(V))$ even becomes a bialgebra. Given a non-commutative k-algebra A, in the light of the previous paragraph, one is inclined to call an algebra $\underline{\mathrm{end}}(A)$ a non-commutative endomorphism ring of A, if there is an algebra map

$$A \to \underline{\mathrm{end}}(A) \otimes A,$$

such that $\underline{\mathrm{end}}(A)$ is universal among all algebras B, with algebra map $A \to B \otimes A$. The bialgebra structure then comes for free. These are the bialgebras studied in the article [51], for 'nice' families of algebras A, which we will explain a little further down.

Abstract motivation aside, bialgebras of the type mentioned above have been appearing recently among combinatorists and knot theorists, studying such things as colored Jones polynomials and the quantum MacMahon master theorem. Our interest lies elsewhere: it is easy to see that even in the commutative case, for example $A = SV$, V of dimension 2, the bialgebra $\underline{\mathrm{end}}(A)$ is not equal to $\mathcal{O}(\mathrm{End}(V)) = \mathcal{O}(M_2)$, the coordinate ring of 2×2 matrices over k, but is quite a bit larger:

$$\underline{\mathrm{end}}(SV) = k\langle a, b, c, d\rangle \Big/ \begin{pmatrix} ac - ca \\ ad - cb + bc - da \\ bd - db \end{pmatrix}$$

Our goal in this thesis then, is to justify why this bialgebra can be considered a non-commutative coordinate ring of the variety under consideration in the simplest case: $A = SV$, V of dimension n. It is our hope that this is no lotus eating, and that the $\underline{\mathrm{end}}$-construction can actually help, both in studying the *geometry* of (commutative) varieties, and in studying non-commutative algebras *geometrically*. Its usefulness derives from its universal, and thus functorial nature. Since modern geometry seems to be taking place more and more in a derived setting, an overarching motivation for using non-commutative algebras in commutative theories could be that

Derived categories are blind to commutativity

by which we mean that $D^b(-)$ is invariant under Morita equivalence.

Let us look at the contents. In Chapter 2, we introduce the main type of algebra we are interested in: Koszul algebras. After a quick look at their origin, we prove some basic results about them, as explicitly as possible. The chapter ends with a survey of examples and applications, taken mostly from geometry and representation theory. Chapters 3 and 4 represent the main body of the paper [51]. In Chapter 3, we introduce distributive algebras as generalizations of Koszul algebras, and show they are deeply connected to a certain monoidal category of quiver representations. Chapter 4 elaborates on the $\underline{\mathrm{end}}(A)$ construction we introduced in the previous paragraph, and shows how one can obtain the full representation theory of this bialgebra in the case when A is distributive. Chapters 5 and 6 represent an attempt to motivate why the notations $\mathcal{O}_{nc}(M_n) := \underline{\mathrm{end}}(k[x_1, \ldots, x_n])$ and $\mathcal{O}_{nc}(GL_2) := H(\underline{\mathrm{end}}(k[x, y]))$ are justified, H denoting 'Hopfication'. Chapter 5 contains an exposition of the representation theories of the symmetric and general linear groups, and applies this to $A = k[x_1, \ldots, x_n]$. Chapter 6 is original: first we establish a link between $\mathcal{O}_{nc}(M_n)$ and the theory of non-commutative symmetric functions. Then we explain some incomplete ideas about a possible connection to Tannaka-Krein duality. Last, we show how completely different techniques are required to study $\mathcal{O}_{nc}(GL_2)$, though these have their roots in the commutative theory as well (in this case: reductive algebraic groups). Let me note here that many of the proofs and ideas in this last chapter originated with my promotor Michel Van den Bergh, and my role has definitely been second to his.

To end this introduction, an apology. The reader who aspects minute proofs of everything he reads, will be disappointed. All material corresponding to the paper [51] itself is completely proven, often in greater detail than the original. Parts that discuss examples or applications however, are often no more than a sketch of what one can find in the references. The reason for this is two-fold: on the one hand, the areas of mathematics touched upon are often so diverse that including all details would make this document even longer than it already is. Why not omit them? This is the second point: I enjoy seeing the connections among concepts in different branches, and cannot refrain from mentioning them, even if only in passing. To compensate for this defect, I have tried including as many references as possible, always being clear as to what exactly can be found in them, so as not to inundate the reader. Also, in some cases, simple proofs of more elementary statements reflecting the real proofs are given; I hope this will suffice. In conclusion, I should apologize for the incomplete nature of Chapter 6: I wanted to include some of the material discussed with Van den Bergh so as to provide an idea of the possible usefulness and ubiquity of the $\underline{\mathrm{end}}$ construction. The most exciting idea, a possible connection between the representations of $\mathcal{O}_{nc}(GL_3)$ and solutions of the Markoff equation in number theory, has been left out. Whether this idea, and the other ones presented in Chapter 6 can be made into rigorous theories, will have to be decided in the future.

2

Koszul algebras

In this chapter we will consider the main types of algebra encountered in the article by B. Kriegk and M. Van den Bergh. Koszul algebras grew out of the study of Koszul complexes in Lie algebra cohomology [18] by Jean-Louis Koszul and in commutative algebra [15] by Arthur Cayley. After a small introduction as to these origins, we will motivate the definition of Koszul algebras by linking it with global dimension in the setting of connected algebras. Subsequently, the main properties of (not necessarily commutative) Koszul algebras will be reviewed, and we end by giving a large number of examples and applications. This last part should be considered as a cursory survey of Koszulity throughout mathematics: complete proofs are not given. In most cases we have tried to present an elementary form of the arguments required, and ample references are provided in case the reader craves a more thorough treatment.

2.1 Origins

Before diving into the general theory, let us give an informal overview of the origins of Koszul algebras in Lie theory and commutative algebra. The algebras themselves were introduced by Priddy [67] only in 1970, in an attempt to abstract the properties of the so called Koszul complex. While studying the de Rham cohomology of a compact semisimple Lie group G, Cartan introduced a cohomology theory for Lie algebras. The idea is that one can obtain from a smooth differential form on G a G-invariant one by an averaging process very similar to the one used in the classical proof of Maschke's theorem. In this case the finiteness of the group is replaced by compactness, and the averaging process is obtained by the existence of a suitable invariant measure on G, known as the Haar measure. One then shows that the inclusion of equivariant differential forms inside all smooth differential forms is homotopic to the identity map, providing an isomorphism in cohomology. This means that to study the de Rham theory of G, one only has to dualize elements in the Lie algebra $\mathfrak{g}$, which consist of (say) right-invariant vector fields on G. These elements obviously correspond to alternating multilinear forms on $\mathfrak{g}$, and fit into a complex. The complex thus obtained is the Koszul complex, and was used by Koszul to compute this cohomology. Slightly more generally, if we take M to be a right $\mathfrak{g}$-module, then the homology of $\mathfrak{g}$ with coefficients in M can be seen to be computed as the homology of the complex:

$$\cdots \to M \otimes_{U\mathfrak{g}} U\mathfrak{g} \otimes_k \Lambda^n \mathfrak{g} \to M \otimes_{U\mathfrak{g}} U\mathfrak{g} \otimes_k \Lambda^{n-1} \mathfrak{g} \to M \otimes_{U\mathfrak{g}} U\mathfrak{g} \otimes_k \Lambda^{n-2} \mathfrak{g} \to \cdots$$

with a suitable differential. This complex (or its dual) is Koszul's original one, even though it now goes under the name of Chevalley-Eilenberg complex. The good properties of this resolution will later be seen as a consequence of the fact that $U\mathfrak{g}$ and $\wedge^*\mathfrak{g}$ are Koszul dual.

From the perspective of commutative and homological algebra, Koszul complexes are used to study so called regular sequences, which provide an extension to the notion of nonzerodivisor. The following is based on Eisenbud [24]. In this context one should mention the Koszul complex's importance to the algebraic notion of depth, which parallels the more geometric notion of codimension. It is also of vital importance for the study of the Cohen-Macaulay condition. For the rest of this paragraph, R will always be assumed to be a commutative, Noetherian ring.

Definition 2.1.1 *Given an R-module M, and a sequence of elements $x_1, \ldots, x_n \in R$, this sequence is called regular on M if $(x_1, \ldots, x_n)M \neq M$, and for $i \in \{1, \ldots, n\}$, x_i is a nonzerodivisor on $M/(x_1, \ldots, x_{i-1})M$.*

Koszul complexes of length 1 and 2 may be defined intuitively as:

$$K(x) : 0 \longrightarrow R \xrightarrow{\cdot x} R$$

$$K(x, y) : 0 \longrightarrow R \xrightarrow{f} R \oplus R \xrightarrow{g} R,$$

where $f(r) = (yr, xr)$, and $g(r, s) = -xr + ys$. Using this, one proves the following theorem:

Theorem 2.1.2 *If R is Noetherian, local, and x and y are contained in the maximal ideal, then the sequence x, y is regular if and only if $H^1(K(x, y)) = 0$.*

Proof: See Eisenbud [24], Chapter 17, §17.1. □

Not only does this mean one can study (short) regular sequences by homological algebra, but it also has as a corollary that permutations of regular sequences are still regular sequences, which isn't immediately obvious from the definition. More generally, given a Noetherian commutative ring R, a free R-module N of rank n, and an element $x = (x_1, ..., x_n) \in R^n \cong N$, the Koszul complex $K(x)$ is defined to be the complex:

$$0 \to R \to N \to \wedge^2 N \to \cdots \to \wedge^i N \to \cdots$$

with differential d_x defined by $d_x(a) = x \wedge a$. In this case, the homology of the Koszul complex detects the lengths of maximal regular sequences in the ideal $(x_1, \ldots, x_n)$.

Theorem 2.1.3 *Let M be a finitely generated module over R, and suppose*

$$H^j(M \otimes K(x_1, \ldots, x_n)) = 0, \text{ for } j < r$$

while

$$H^r(M \otimes K(x_1, \ldots, x_n)) \neq 0$$

then every maximal regular sequence on M in $I = (x_1, \ldots, x_n)$ has length r.

Corollary 2.1.4 *If $x_1, \ldots, x_n$ is a regular sequence on M, then $M \otimes K(x_1, \ldots x_n)$ is exact except at the n-th spot, where the homology is equal to $M/(x_1, \ldots, x_n)M$.*

Proof: See Eisenbud [24], Chapter 17, §17.2. □

This theorem and its corollary are used as the foundation to study the notion of depth. For $M \neq IM$, the lengths of all maximal regular sequences in I on M are the same; this length is called the depth of I on M. For local rings, this is only the beginning of a beautiful theory developed by Auslander, Serre and Buchsbaum. The depth of I (this is for $M = R$) is an arithmetic measure of the size of I, which depends only on the radical of I, and is thus geometric in nature. Rings for which the depth of every ideal is equal to its codimension are called Cohen-Macaulay, and they correspond geometrically to affine schemes having many nice properties in common with non-singular varieties.

Suppose V is a finitely generated free module over a commutative ring k. If $\{e_i\}_i$ is a basis of V, set

$$t = \sum_i e_i^* \otimes e_i \in V^* \otimes V = \wedge^1(V^*) \otimes S_1(V),$$

so we can regard t as an element of the algebra $\wedge(V^*) \otimes S(V)$. In this algebra,

$$t^2 = \sum_{i,j} (e_i^* \wedge e_j^*) \otimes (e_i \cdot e_j) = 0,$$

so there is a complex of free $S(V)$-modules

$$K : S(V) \to V^* \otimes S(V) \to \cdots \to \wedge^d(V^*) \otimes S(V) \to \cdots,$$

maps given by multiplication by t. This complex is called the tautological Koszul complex, and is naturally isomorphic to the Koszul complex $K(t)$, if we regard t as an element of the $S(V)$-module $V^* \otimes S(V)$. This complex appears naturally all throughout commutative algebra. What Priddy did was introduce a generalized tautological Koszul complex associated to any commutative algebra A with quadratic relations (recovering the original one for $A = S(V)$). If this complex is exact, we speak of a Koszul algebra. In Section 2.3, we will continue this story in the more general setting of non-commutative algebras.

2.2 The global dimension of connected algebras

To get accustomed to the definition of a Koszul algebra, and the language it is presented in, we follow the approach given in Krähmer [50], by proving the following theorem, which very naturally leads to Koszul algebras:

Theorem 2.2.1 *Let $A = \bigoplus_{i \geq 0} A_i$ be a Noetherian, graded algebra over a field $k = A_0$. Then we have*

$$\mathrm{gldim}(A) = \mathrm{pd}(k) = \sup\{i \mid \mathrm{Ext}_A^i(k,k) \neq 0\} = \sup\{i \mid \mathrm{Tor}_i^A(k,k) \neq 0\}$$

This theorem is also of independent interest as a means of calculating the global dimension of a big class of Noetherian rings. From now on, an algebra is always considered to be a unital, $\mathbb{N}$-graded, locally finite, associative k-algebra, that is connected, meaning that $A_0 = k$. Usually the algebras will also be considered to be left and right Noetherian. Unless explicitly stated otherwise, all modules will be left modules. For graded k-vector spaces V and W, we put

$$\underline{\mathrm{Hom}}_k(V,W) = \bigoplus_{j \in \mathbb{Z}} \mathrm{Hom}_k^j(V,W),$$

where $\mathrm{Hom}_k^j(V,W) = \{\phi \in \mathrm{Hom}_k(V,W) \mid \phi(V_i) \subset W_{i-j}\}$. Morphisms of graded vector spaces are elements of Hom_k^0. Modules over these algebras will be taken to be graded modules, and morphisms will again be considered degree-preserving. Notice that the ground field k has a trivial structure of both left and right A-module by the augmentation map $\epsilon : A \to A/A_+ \cong k$. Sometimes tensor products will be graded, which means that for graded A-modules M and N, $(M \otimes N)_i = \bigoplus_{r+s=i} M_r \otimes N_s$, and for the graded tensor product of linear maps we have $(\phi \otimes \psi)(m \otimes n) = (-1)^{|\psi||m|}\phi(m) \otimes \psi(n)$, where $|\psi| = j$ if $\psi \in \mathrm{Hom}_k^j(M,N)$. This product will be denoted $\underline{\otimes}$ if we want to emphasize the difference, which is of importance only for the product of maps. There are a number of homological consequences that follow from the grading. The most important proposition in this context seems to be the graded version of Nakayama's lemma:

Proposition 2.2.2 (Nakayama's lemma) *For M a finitely generated, graded (=fgg) A-module we have*

$$M = 0 \text{ iff } k \otimes_A M = M/A_+M = 0$$

Proof: First note that a finitely generated graded A-module has to be bounded below, by the minimum degree of the generators, say s. Then $A_+M = \bigoplus_{j>s} M_j$, which is not equal to M, unless $M = 0$. Finally, note that $k \otimes_A M = A/A_+ \otimes_A M = M/A_+M$. □

Notice that not only the graded assumption but also the finite generation is crucial: the ring $k[x, x^{-1}]$ is a module over $k[x]$ in the obvious way, though $k \otimes_{k[x]} k[x, x^{-1}] = 0$. Nakayama's lemma allows us to prove the following crucial property of graded projective modules.

Proposition 2.2.3 *For M a finitely generated graded A-module, the following are equivalent:*

1. *M is flat*
2. *M is projective*
3. *M is free*

Proof: Obviously, free $\Rightarrow$ projective $\Rightarrow$ flat. If M is flat, then in particular $\mathrm{Tor}_1^A(k, M) = 0$ (note that we view k as a right module here). As a graded vector space, write $M = V \oplus A_+M$, for some finitely generated graded k-module V and define an epimorphism:

$$\phi : A \otimes V \to AV \subset M : a \otimes x \mapsto ax$$

Note that the tensor product on the domain is a tensor product of graded vector spaces. Now

$$\frac{M}{AV} \Big/ \frac{A_+M}{AV} = 0,$$

so Nakayama implies that $M = AV$, and the induced morphism $\tilde{\phi} : V = k \otimes_A A \otimes V \to k \otimes_A M$ is an isomorphism of finitely generated graded k-modules. Now ϕ gives a short exact sequence:

$$0 \to \mathrm{Ker}\ \phi \to A \otimes V \to M \to 0,$$

and thus an exact sequence

$$\cdots \to \mathrm{Tor}_1^A(k, M) \to k \otimes_A \mathrm{Ker}\ \phi \to k \otimes_A A \otimes V \to k \otimes_A M \to 0,$$

which is short because of our assumption. This means $k \otimes_A \text{Ker } \phi$ is a kernel to $\tilde{\phi}$, which is an isomorphism, and Nakayama then implies that Ker $\phi = 0$, so M is free. $\square$

A fgg A-module M is free in the category of fgg A-modules if it is of the form $A \underline{\otimes} V \cong A^n$ for some finite dimensional graded vector space V. Then, M is also free in the categories of (not necessarily fg) graded A-modules, fg (not necessarily graded) A-modules, and A-modules. The proofs show that the proposition is true for M a fgg A-module independent of which category we're working in. Passing from $\underline{\otimes}$ to $\otimes$ corresponds to forgetting the grading, which transfers to derived functors. In the same vein,

Proposition 2.2.4 *For M a fgg A-module, and N a graded A-module we have*

$$\underline{\text{Hom}}_A(M, N) = \text{Hom}_A(M, N)$$

Proof: Given homogeneous generators $m_1, \ldots, m_n$ of M, then for $\phi \in \text{Hom}_A(M, N)$, one has $\phi(m_i) \in \bigoplus_{j=s_i}^{r_i} N_j$, for some finite s_i, r_i. For $s = \min\{|m_i| - s_i\}$, and $r = \max\{|m_i| - r_i\}$, we then have

$$\phi \in \bigoplus_{j=s}^{r} \text{Hom}_A^j(M, N).$$

$\square$

This also means that we do not have to distinguish between Ext_A and $\underline{\text{Ext}}_A$ when working with fgg A-modules. Now we will consider minimal resolutions. Since in our case every projective module is free, a projective resolution of such a module will be completely determined by a number of matrices. Let's fix notation: a morphism between free modules $A^{b_i} \to A^{b_{i-1}}$ is given by a matrix $T_i \in M_{b_i \times b_{i-1}}(A)$. This allows for the following definition:

Definition 2.2.5 *A (graded) projective resolution of a finitely generated, graded A-module is called minimal if $T_i \in M_{b_i \times b_{i-1}}(A_+)$ for all i.*

The justification of the terminology will come in due course. A priori it is not clear that these resolutions have to exist, but one has the following

Proposition 2.2.6 *For a Noetherian algebra A, any fgg A-module admits a minimal, projective resolution (as fgg module).*

Proof: Given a fgg module M, construct an epimorphism $\phi : A^{b_0} \to M$ just like in Proposition 2.2.3. Noetherianity now ensures that Ker ϕ is again a fgg A-module, and we can repeat this process to get a resolution $(\phi_i)_i$ of M. The same proposition now shows that $\phi_i \otimes k$ is an isomorphism. Obviously then, the kernel of ϕ_i is contained in $A_+^{b_i}$, for otherwise $\phi_i \otimes_A k$ would not be injective (remember that k is an A-module by $k \cong A/A_+$). $\square$

Minimality implies that $\text{Ext}_A(M, k)$ and $\text{Tor}^A(k, M)$ can be easily computed. Since every T_i has entries in A_+, and morphisms are always considered to preserve the degree, having a minimal resolution:

$$P_\bullet^{\min} = \cdots \to A^{b_1} \to A^{b_0} \to 0$$

just means that the boundary maps of the complexes

$$\mathrm{Hom}_A(P_\bullet^{\min}, k) = 0 \to \mathrm{Hom}_A(A^{b_0}, k) \to \mathrm{Hom}_A(A^{b_1}, k) \to \cdots$$

and

$$k \otimes_A P_\bullet^{\min} = \cdots \to k \otimes_A A^{b_1} \to k \otimes_A A^{b_0} \to 0$$

are zero! This implies that the (co)homology of the complexes is given by the complexes themselves, and since $\mathrm{Hom}_A(A^{b_i}, k) \cong k \otimes_A A^{b_i} \cong k^{b_i}$ as k-spaces,

$$\mathrm{Ext}_A^i(M, k) \cong \mathrm{Tor}_i^A(k, M) \cong k^{b_i}$$

Since we know that the Ext and Tor functors don't depend on the particular projective resolution chosen, this immediately implies the following

Corollary 2.2.7 *The ranks b_i in a minimal resolution are uniquely determined by the module M. In particular, the length of a minimal resolution is unique, and the minimum number of generators of M is given by b_0. In the case $M = k$, this implies that $\epsilon : A \to k$ is the first step in its minimal resolution.*

This allows for a justification of terminology.

Proposition 2.2.8 *For a fgg A-module M, the length l of a minimal resolution is equal to* $\mathrm{pd}(M)$.

Proof: Obviously, $\mathrm{pd}(M) \leq l$. If there would be a resolution of smaller length, then $\mathrm{Ext}_A^l(M, k) = \mathrm{Tor}_l^A(k, M) = 0$, which contradicts the previous statements. □

Using this, part of the theorem that was stated at the beginning of this section is clear:

Corollary 2.2.9 *For M a fgg A-module,*

$$\mathrm{pd}(M) = \sup\{i \mid \mathrm{Ext}_A^i(M, k) \neq 0\} = \sup\{i \mid \mathrm{Tor}_i^A(k, M) \neq 0\}$$

To prove the other equalities in the theorem, let's look at the case $M = k$. By repeating this whole paragraph for right modules and remembering that $\mathrm{Tor}(P, Q)$ can be computed both from a projective resolution of Q and the functor $P \otimes_A -$ or from a projective resolution of P and the functor $- \otimes_A Q$, one sees that the projective dimensions of k as a left and as a right A-module coincide. This means there's a minimal resolution of the right A-module k of length $\mathrm{pd}(k)$, which can be used to compute $\mathrm{Tor}^A(k, M)$, obtaining the result

Proposition 2.2.10 *For all fgg A-modules M,* $\mathrm{pd}(M) \leq \mathrm{pd}(k)$.

Remark 2.2.11 *This approach draws heavily on the fact that our rings are Noetherian, connected and graded. This type of method does not work well for general rings. A similar theory can be and has been however developed for commutative local rings [24]. For arbitrary rings one does have a theory of minimal injective resolutions based on injective hulls, though this will not concern us here.*

To illustrate the theory, we'll construct a minimal resolution of k for the quantum plane. The calculations of this example will be typical for Koszul algebras which we'll introduce in the next section.

Example Remember that $A = k\langle x, y\rangle/(xy - qyx)$. First off, we'll look at the structure of the quantum plane. This turns out to be familiar, and is contained in the following easy

Lemma 2.2.12 *Define the automorphism $\alpha : k[x] \to k[x] : x \to qx$. Then A is isomorphic to the Ore extension $k[x][y, \alpha, 0]$. As a consequence it's Noetherian and has no zero-divisors (see Kassel [49], Chapter 1, §1.7). A basis for the underlying vector space is given by $\{x^i y^j\}_{i,j\geq 0}$.*

To construct a resolution for k, remember that the first term is always given by $\epsilon : A \to k$. The kernel of this augmentation mapping is obviously generated by x and y, so put $P_1^{\min} = A^2$ and

$$\phi_1 : A^2 \to A : (f, g) \mapsto fx + gy.$$

To determine the kernel of ϕ_1, we use the basis $\{x^i y^j\}_{i,j\geq 0}$, then make the following computation:

$$\begin{aligned}\phi_1(\sum_{i,j} a_{ij} x^i y^j, \sum_{r,s} b_{rs} x^r y^s) &= \sum_{i,j} a_{ij} q^{-j} x^{i+1} y^j + \sum_{r,s} b_{rs} x^r y^{s+1} \\ &= \sum_{i\geq 0} (a_{i0} x^{i+1} + b_{0i} y^{i+1}) + \sum_{i,j>0} (a_{(i-1)j} q^{-j} + b_{i(j-1)}) x^i y^j\end{aligned}$$

From this we deduce that $a_{i0} = 0 = b_{0i}$ and $a_{i(j+1)} q^{-j-1} + b_{(i+1)j} = 0$. This means the kernel is generated as an A-module by $(-qy, x)$. Accordingly, put $\phi_2 : A \to A^2 : f \mapsto (-qfy, fx)$. Since A is a domain, this morphism has trivial kernel, and we have constructed a resolution. In conclusion, $\mathrm{pd}(k) = 2$.

Remark 2.2.13 *In general, it is rather hard to determine when a set of monomials is a k-basis for an algebra. Later on, we'll see an (in principle) easier way to do this, using Bergman's diamond lemma.*

To prove Theorem 2.2.1, we'll need some more information about non-graded finitely generated A-modules, since the global dimension takes these into account as well. This will require a

Definition 2.2.14 *An algebra A over a field k is called filtered if there exists an increasing sequence $\{0\} \subset F_0 \subset F_1 \subset \cdots \subset A$ of subspaces, such that $1 \in F_0, A = \cup_{n\in\mathbb{N}} F_n$, and $F_i \cdot F_j \subset F_{i+j}$. If A is filtered, the associated graded algebra B is defined by $B_j = F_j A / F_{j-1} A$ and $B_0 = F_0 A$.*

A graded algebra A provides a trivial example (with associated graded A itself), which only becomes interesting by passing to fg A-modules M. Taking $m_1, \ldots, m_n$ as generators, M becomes a filtered module by $F_j M = \{a_1 m_1 + \ldots + a_n m_n \mid a_i \in F_j A\}$, and $(F_j A)(F_k M) \subset F_{j+k} M$. The associated graded module $N = \bigoplus_j F_j M / F_{j-1} M$ then becomes a module over the associated graded ring, in this case A itself. The main result in this theory says that a graded free resolution of an associated graded module can be lifted to a projective resolution of the module itself, respecting the length. Details can be found in McConnell and Robson [58], Chapter 7, §6. This theorem immediately implies that $\mathrm{pd} M \leq \mathrm{pd} k$ for any finitely generated A-module, not just the graded ones. Remember that the global dimension is defined as

$$\mathrm{gldim}(A) = \sup\{\mathrm{pd}(M) \mid M \text{ an } A\text{-module}\} = \sup\{d \mid \mathrm{Ext}_A^d(M, N) \neq 0, \text{for some } M, N \ A\text{-modules}\}$$

Even though we only have results for fg modules, it turns out that to calculate the global dimension, one need only consider cyclic modules, see Weibel [80], Chapter 4, §1 for a proof. However, in the resolutions themselves, modules can still fail to have finite generation, so it is here that we play on the safe side by assuming A to be Noetherian. This hypothesis also allows for the equal treatment of left and right global dimension, which again becomes a problem in the non-Noetherian case. Finally, we have proved the theorem that we started with:

Theorem 2.2.15 *Let $A = \bigoplus_{i\geq 0} A_i$ be a Noetherian, graded algebra over a field $k = A_0$. Then we have*

$$\mathrm{gldim}(A) = \mathrm{pd}(k) = \sup\{i \mid \mathrm{Ext}^i_A(k,k) \neq 0\} = \sup\{i \mid \mathrm{Tor}^A_i(k,k) \neq 0\}$$

The use of these equalities lies in the fact that many theorems throughout commutative and non-commutative algebra require finite global dimension, which is in general hard to check because of the universal quantifier. The aforementioned results now provide us with a possible algorithm: construct the minimal resolution of k.

2.3 Koszul algebras

Though the theorem we proved in the previous paragraph provides a nice algorithm in principle, in practice it can still be very hard to construct a minimal resolution. A very large class of algebras for which this can be achieved easily, is the class of Koszul algebras. The material in this section is based on the book by Polishchuk and Positselski [66].

Definition 2.3.1 *A unital, $\mathbb{N}$-graded, locally finite, associative k-algebra A is a Koszul algebra if there exists a minimal linear resolution of k; this means that the entries making up the matrices T_i of this resolution are contained in A_1.*

Remark 2.3.2 *As we explained in the previous section, arguments concerning the categories we're working in are unnecessary if A is assumed to be Noetherian, which is what we will generally assume.*

Examples will come later, though it's already obvious that the quantum plane (and by setting $q = 1$, the ring $k[x, y]$) is one of them. To any Koszul algebra A, one can associate its Koszul dual $A^!$, and a large part of their appeal stems from the power of this duality. Passing over to the Koszul dual can have many different effects: it can mean passing from the infinite dimensional to the finite dimensional, the non-commutative to the commutative, ... To show how this works, we need some general structure theory. Using Proposition 2.2.4, there is a decomposition of $\mathrm{Hom}_A(M, N)$ into grading components, which carries over to the homology level:

$$\mathrm{Ext}^i_A(k,k) = \bigoplus_{j\in\mathbb{Z}} \mathrm{Ext}^{ij}_A(k,k).$$

This simple observation allows for a first equivalent description of Koszulity

Proposition 2.3.3 *For $j < i$, we have $\mathrm{Ext}^{ij}_A(k,k) = 0$. The algebra A is Koszul if and only if $\mathrm{Ext}^{ij}_A(k,k) = 0$ for $j > i$, and so for the full Ext-space (later we'll show this is an algebra) we have*

$$\mathrm{Ext}_A(k,k) = \bigoplus_{i\geq 0} \mathrm{Ext}^{ii}_A(k,k)$$

Proof: Use a minimal resolution $P^{\min}_\bullet$ to compute $\mathrm{Ext}_A(k,k)$. Since $1 \in A = P_0$ has degree 0, the maps preserve degree, and the matrices of the resolutions have entries in A_+, P_i has to be generated in degree $\geq i$. If $\mathrm{Hom}^j_A(P_i, k) \neq 0$, and remembering that k is non-trivial only in degree 0, this means that $j \geq i$. This proves the first part. If A is Koszul, then by the same kind of reasoning P_i has to be generated in degree i. For $j > i$, a non-zero $\phi \in \mathrm{Hom}^j_A(P_i, k)$ would have to map elements of degree j in P_i to k, and the rest to zero. This would mean the generators get mapped to zero, and the statement is proven. $\square$

The following result puts a severe constraint on Koszul algebras, which is one of the reasons we will want to define higher Koszul algebras later on. We'll first need a definition.

Definition 2.3.4 *A connected, graded associative algebra A over a field k is quadratic if it is of the form $TV/(R)$, where TV is the tensor algebra on a finite dimensional vector space, and R is a subspace of $V \otimes V$.*

The symmetric and exterior algebra are obvious examples of this, as is the quantum plane $k\langle x, y\rangle/(xy - qyx)$, for $q \in k^*$.

Proposition 2.3.5 *Any Koszul algebra is quadratic.*

Proof: Just look at the first few terms of a linear resolution of k:

$$\cdots \to A^{b_2} \xrightarrow{\phi_2} A^{b_1} \xrightarrow{\phi_1} A \xrightarrow{\epsilon} 0.$$

We know that ϕ_1 is given by $T_1 \in M_{b_1 \times 1}(A_1)$. Since Ker $\epsilon = A_+ =$ Im ϕ_1, A_+ is generated as an A-module by A_1, and thus also A is generated as an algebra by A_1, and $b_1 = \dim_k(A_1)$. Since A^{b_2} is generated in degree 2, and Im $\phi_2 =$ Ker ϕ_1 (which are just the relations amongst the generators), these relations have to be quadratic. $\square$

Remark 2.3.6 *The converse implication is not true since continuing this kind of reasoning will impose constraints on the higher degrees.*

For any quadratic algebra A, one can introduce the dual algebra $A^!$. Since Koszul algebras are quadratic, we can talk about the Koszul dual.

Definition 2.3.7 *If A is Koszul, then it can be written as $A = TV/(R)$, for some subspace $R \subset V \otimes V$. Set*

$$A^! = TV^*/(R^\perp), \text{ where } R^\perp = \{r \in V^* \otimes V^* \mid r(R) = 0\}$$

Example 2.3.8 *For the quantum plane $A = k\langle x, y\rangle/(xy - qyx)$,*

$$\begin{aligned}
V &= kx + ky \\
R &= k(x \otimes y - qy \otimes x) \\
V^* \otimes V^* &= kx^* \otimes x^* + kx^* \otimes y^* + ky^* \otimes x^* + ky^* \otimes y^* \\
R^\perp &= kx^* \otimes x^* + k(x^* \otimes y^* + q^{-1}y^* \otimes x^*) + ky^* \otimes y^* \\
A^! &= k\langle w, z\rangle/(w^2, wz + q^{-1}zw, z^2)
\end{aligned}$$

Notice that the Koszul dual of the commutative, infinite dimensional polynomial ring $k[x, y]$ becomes the non-commutative, finite dimensional exterior algebra $\wedge V$. To continue the discussion, we'll need a product on $\mathrm{Ext}_A(k, k)$, to make it into an algebra. First off, we'll need to say something about the bar resolution of a connected graded algebra, defined as (terms as graded A-modules in the obvious way):

$$\cdots \to A \otimes A_+ \otimes A_+ \xrightarrow{\phi_2} A \otimes A_+ \xrightarrow{\phi_1} A \xrightarrow{\epsilon} k,$$

with boundary maps given by

$$\phi_i : a_0 \otimes \cdots \otimes a_i \mapsto a_0a_1 \otimes \cdots \otimes a_i - a_0 \otimes a_1a_2 \otimes \cdots \otimes a_i + \ldots + (-1)^{i-1}a_0 \otimes \cdots \otimes a_{i-1}a_i$$

Using this we prove

Proposition 2.3.9 *The bar sequence $C_\bullet$, where $C_i = A \otimes A_+^{\otimes i}$, is a graded free resolution of the ground field k.*

Proof: That the terms in the complex are graded free A-modules is obvious. That $d^2 = 0$ follows inductively from computations like:

$$\begin{aligned} a_0 \otimes a_1 \otimes a_2 \otimes a_3 &\mapsto a_0a_1 \otimes a_2 \otimes a_3 - a_0 \otimes a_1a_2 \otimes a_3 + a_0 \otimes a_1 \otimes a_2a_3 \\ &\mapsto a_0a_1a_2 \otimes a_3 - a_0a_1 \otimes a_2a_3 - a_0a_1a_2 \otimes a_3 + a_0 \otimes a_1a_2a_3 + a_0a_1 \otimes a_2a_3 - a_0 \otimes a_1a_2a_3 \\ &= 0 \end{aligned}$$

The differential also obviously respects the total grading. If $a_0 \otimes \cdots \otimes a_i$ gets mapped to 0, then it can be written as the image of $1 \otimes (a_0 - \epsilon(a_0)) \otimes \cdots \otimes a_i$, and since $\mathrm{Im}(A \otimes A_+ \to A) = A_+$, this is obviously a resolution of k. $\square$

This resolution is called the normalised bar resolution of the fgg A-module k, and allows us to view $\mathrm{Ext}_A(k,k)$ as the cohomology of the cochain complex $\mathrm{Hom}_A(C_\bullet, k)$. There is an obvious isomorphism of vector spaces $(A_+^{\otimes i})^* \cong \mathrm{Hom}_A(C_i, k)$ by pre-tensoring with ϵ, and the transported differential is given by

$$d : (A_+^{\otimes i})^* \to (A_+^{\otimes i+1})^* : \phi \mapsto d\phi,$$

$$d\phi(a_1 \otimes \cdots \otimes a_{i+1}) = -\phi(a_1a_2 \otimes \cdots \otimes a_{i+1}) + \ldots + (-1)^i \phi(a_1 \otimes \cdots \otimes a_ia_{i+1}).$$

From now on, denote $(A_+^{\otimes i})^*$ by C^i. It is then obvious that the graded Leibniz rule holds:

$$d(\phi \otimes \psi) = d\phi \otimes \psi + (-1)^{|\phi|} \phi \otimes d\psi,$$

using the definition

$$(\phi \otimes \psi)(a_1 \otimes \cdots \otimes a_{i+j}) = \phi(a_1 \otimes \cdots \otimes a_i)\psi(a_{i+1} \otimes \cdots \otimes a_j), \tag{2.1}$$

turning $(C^\bullet, d)$ into a DG-algebra. The basic fact about DG-algebras is that their cohomology has a graded ring structure, by

$$[\phi] \cdot [\psi] := [\phi \otimes \psi].$$

In this case the cohomology corresponds to $\mathrm{Ext}_A(k,k)$, called the Yoneda algebra from now on.

Remark 2.3.10 *Given a finite dimensional algebra A, this Yoneda algebra contains a lot of information about the representations of A. Indeed, given a finitely generated A-module, we know it can be realized by a finite sequence of extensions of simple A-modules.*

The product will be denoted by

$$\smile : \mathrm{Ext}_A^i(k,k) \otimes \mathrm{Ext}_A^j(k,k) \to \mathrm{Ext}_A^{i+j}(k,k),$$

By using the internal grading on $C_\bullet$,

$$(C_i)_j = \bigoplus_{\substack{j_0 + \ldots + j_i = j \\ j_1, \ldots, j_i \geq 1}} A_{j_0} \otimes \cdots \otimes A_{j_i},$$

we get the internal grading on $\mathrm{Ext}^i_A(k,k)$ discussed in Proposition 2.3.3. On C^i, this becomes a direct product

$$C^i = \left(\bigoplus_{j\geq 0}(A_+^{\otimes i})_j\right)^* = \prod_{j\geq 0}(A_+^{\otimes i})^*_j,\ (A_+^{\otimes i})_j = \bigoplus_{\substack{j_1+\ldots+j_i=j\\ j_1,\ldots,j_i\geq 1}} A_{j_1}\otimes\cdots\otimes A_{j_i}$$

Furthermore, the differential d respects this grading: $d(C^{ij})\subset C^{(i+1)j}$, and working in the graded category (see Proposition 2.2.4) this product becomes a sum. In conclusion,

$$\mathrm{Ext}^{ij}_A(k,k)\smile \mathrm{Ext}^{pq}_A(k,k)\subset \mathrm{Ext}^{(i+p)(j+q)}_A(k,k),$$

and we see that $E := \oplus_{i\geq 0}\mathrm{Ext}^{ii}_A(k,k)$ is a subalgebra of $\mathrm{Ext}_A(k,k)$. The connection to quadratic algebras is given by the following

Proposition 2.3.11 *For A a quadratic algebra,*

$$A^! \cong \bigoplus_{i\geq 0}\mathrm{Ext}^{ii}_A(k,k),$$

which is also the subalgebra of $\mathrm{Ext}_A(k,k)$ generated by $\mathrm{Ext}^1_A(k,k)$.

Proof: Since A is locally finite dimensional we have

$$C^{ij} = \bigoplus_{\substack{j_1+\ldots+j_i=j\\ j_1,\ldots,j_i\geq 1}} A^*_{j_1}\otimes\cdots\otimes A^*_{j_i}.$$

From this it easily follows that $dC^{ii}=0$, and consequently $\mathrm{Ext}^{ii}(k,k)=C^{ii}/dC^{(i-1)i}$. Since $C^\bullet$ is generated by C^1, the algebra E is generated by $\mathrm{Ext}^{11}_A(k,k)$. Now all one has to do to obtain $A^!$ is to consider the first few terms

$$\begin{aligned}\mathrm{Ext}^{00}_A(k,k) &= A^*_0\cong k\\ \mathrm{Ext}^{11}_A(k,k) &= A^*_1\\ \mathrm{Ext}^{22}_A(k,k) &= (A^*_1\otimes A^*_1)/dA^*_2\end{aligned}$$

Written out, the last one means that a $\phi\in A^*_1\otimes A^*_1$ is zero if $\phi(a_1\otimes a_2)=-\psi(a_1a_2)$, for some $\psi\in A^*_2$. This is exactly saying that ϕ has to be zero on $R\subset A_1\otimes A_1$. Since we already know that A_1 generates A, $\mathrm{Ext}^1_A(k,k)=\mathrm{Ext}^{11}_A(k,k)$. □

This allows for a reformulation of Proposition 2.3.3 and Theorem 2.2.15.

Corollary 2.3.12 *A quadratic algebra A is Koszul if and only if $A^!\cong\mathrm{Ext}_A(k,k)$.*

Corollary 2.3.13 *A Koszul algebra A has finite global dimension if and only if $A^!$ is finite dimensional as a k-vector space.*

The last corollary shows very nicely how a hard property to check for A can be translated into an easy property to check for $A^!$. Notice that in (2.1), where we defined the product on $C^\bullet$, we did not use the Koszul sign convention, for that would require an extra factor of $(-1)^{|\psi||a_1\otimes\cdots\otimes a_i|}$. By using $\underline{\otimes}$ as a product on the C^{ij}, so

$$\phi\underline{\otimes}\psi = (-1)^{jp}\phi\otimes\psi, \text{ for } \phi\in C^{ij} \text{ and } \phi\in C^{pq},$$

we no longer have that $(C^\bullet, d)$ is a DG-algebra (due to sign issues). However, by defining

$$B_n = \oplus_{j-i=n} C^{ij},$$

we do get that $(B_\bullet, d)$ is a DG-algebra, though this time d lowers the degree. This allows for a reformulation of Proposition 2.3.11 and Proposition 2.3.3.

Corollary 2.3.14 *$H_0(B_\bullet)\cong A^!$, and A is Koszul if and only if $H_n(B_\bullet)=0$ for $n>0$.*

To end this section, we introduce the Koszul complex for any quadratic algebra A, and mention a couple of its properties. This complex will be denoted $K_\bullet(A,k)$ and its degree n part is defined (as vector space) by:

$$K_n(A,k) = A\otimes(A^!)^*_n.$$

Consider A as a right A-module via multiplication, and $(A^!)^*$ as right $A^!$-module via $(\phi b)(c)=\phi(bc)$. These actions turn $K_\bullet(A,k)$ into a right $A\otimes A^!$-module, and the differential is defined by the action of $d=\sum_i e_i\otimes e_i^*$, where the e_i and e_i^* form bases for (respectively) $V=A_1$ and $V^*=A_1^!$:

$$d_K : K_n(A,k)\to K_{n-1}(A,k) : a\otimes\phi\mapsto\sum_i ae_i\otimes\phi e_i^*$$

This does not depend on the chosen basis. Now the element d corresponds to the identity on V, so $d\otimes d$ corresponds to $\mathrm{id}_{V\otimes V}\in A_2\otimes A_2^!$. It is easy to see that $A_2\otimes A_2^!$ can be identified with $\mathrm{Hom}_k(R, V\otimes V/R)$, so $d^2=0=d_K^2$. The resulting complex $(K_\bullet(A,k), d_K)$ is called the Koszul complex of A. For $A=SV$, this is exactly the complex we already discussed in a previous section. The importance comes from the following observation.

Proposition 2.3.15 *$K_\bullet(A,k)$ is acyclic if and only if A is Koszul.*

If the Koszul complex is acyclic, the it is obviously a minimal resolution of k, and $\mathrm{Ext}_A(k,k)\cong A^!$, so A is Koszul by Corollary 2.3.12. The other direction uses Corollary 2.3.14 along with a technical argument linking the Koszul complex and the bar resolution. We will skip this part and refer to Polishchuk and Positselski [66]. A small variation of this complex can be used to compute the Hochschild cohomology of A. The details will be left to the interested reader, but the essential part is that the acyclicity of this variation, denoted $K_\bullet(A^e, A)$, where $A^e = A\otimes A^{op}$ also characterizes Koszulity. We mention this since this allows one to deduce

Proposition 2.3.16 *For any quadratic algebra A, the following are equivalent:*

1. *A is Koszul*
2. *A^{op} is Koszul*
3. *$A^!$ is Koszul*

Proof:[sketch] The fact that A^e appears should indicate that, even though we have not defined $K_\bullet(A^e, A)$, the roles of A and A^{op} are completely symmetric therein, indicating the equivalence of the first two statements. By the universal coefficient theorem (see Weibel [80], Chapter 3, §3.6), the dual $K_\bullet(A,k)^*$ of the Koszul complex is acyclic if and only if $K_\bullet(A,k)$ is. But it is immediately checked that $K_\bullet(A,k)^* \cong K_\bullet((A^!)^{op}, k)$ as chain complexes; this proves the last equivalence. □

A last consequence of Koszulity comes from the theory of Hilbert series. Remember that for a graded vector space $V = \oplus_{k\in\mathbb{Z}} V_k$, its Hilbert series $h_V(z)$ is defined by

$$h_V(z) = \sum_{k\in\mathbb{Z}} \dim_k(V_k) \cdot z^i.$$

The double Poincaré series of a graded module M over a graded algebra A is defined by the formula

$$P_{A,M}(u,z) = \sum_{i,j\in\mathbb{Z}} \dim_k(\mathrm{Ext}_A^{ij}(M,k))u^i z^j.$$

If $M = k$, the trivial A-module, we denote $P_{A,k}(z) = P_A(z)$. These definitions allow for a proof of

Proposition 2.3.17 *For any graded algebra A, the following relation holds:*

$$h_A(z)P_A(-1,z) = 1$$

Proof: Take the minimal free graded resolution of k, say $P_\bullet^{\min}$. From the previous sections we know that $\mathrm{Ext}_A^i(k,k) = (k \otimes P_i)^*$, and thus $h_{P_i}(z) = h_{\mathrm{Ext}_A^i(k,k)}(z)h_A(z)$. Now it suffices to pass over to the Euler characteristic of the complex and we're done. □

Corollary 2.3.18 *For a Koszul algebra A one has*

$$h_A(z)h_{A^!}(-z) = 1$$

Let us now summarize this entire chapter in the following theorem, giving equivalent condition for a connected, Noetherian, graded algebra to be Koszul.

Theorem 2.3.19 *Suppose A is a connected, graded, Noetherian algebra. The following are equivalent:*

1. *A is Koszul, i.e. the minimal free resolution of k is linear*
2. $\mathrm{Ext}_A^{ij}(k,k) = 0$ *for $i \neq j$*
3. $\mathrm{Tor}_{ij}^A(k,k) = 0$ *for $i \neq j$*
4. *Both A and the algebra $\mathrm{Ext}_A(k,k)$ are generated in degree 1*
5. $A^! \cong \mathrm{Ext}_A(k,k)$
6. *A is quadratic and $A^!$ is Koszul*
7. *A^{op} is Koszul*

8. *$H_n(B_\bullet) = 0$ for $n > 0$*

9. *The Koszul complex $K_\bullet(A, k)$ is acyclic*

10. *The Koszul complex $K_\bullet(A^e, A)$ (not introduced) is acyclic*

and as a consequence, we have the following useful numerical relation:

$$h_A(z)P_A(-1, z) = h_A(z)h_{A^!}(-z) = 1$$

Remark 2.3.20 *A lot of references wrongly state that this last relation characterizes Koszulity. This was unknown for some time but eventually a counterexample was found. More remarkably, Piontkovskii has proved a theorem saying it is impossible to decide whether a quadratic algebra A is Koszul, knowing only the Hilbert series of A and its dual. This theorem is described in Polishchuk and Positselski [66], Chapter* 3, §5.

2.4 Examples and applications

After this technical section, we present some examples of Koszul algebras and Koszul duality in mathematics, especially in geometry and representation theory. The choice of these examples has been dictated by personal preference, and we do not in any way claim to be exhaustive. Also, since rather diverse techniques are required, proofs are not included, though sometimes a simplified argument will be presented. Some of these examples can be found in Polishchuk and Positselski [66]; the others are taken from the original papers and we give extensive references.

Examples

- As we already mentioned, the commutative polynomial algebra $k[x_1, \ldots, x_n]$ is Koszul. The minimal free resolution of k is given by

$$0 \longrightarrow A^{\binom{n}{n}} \xrightarrow{\phi_n} \cdots \longrightarrow A^{\binom{n}{2}} \xrightarrow{\phi_2} A^{\binom{n}{1}} \xrightarrow{\phi_1} A \longrightarrow k \longrightarrow 0$$

  Taking as basis for $A^{\binom{n}{i}}$ the set $\{e_{m_1 \cdots m_i} \mid 1 \leq m_1 \leq \cdots \leq m_i \leq n\}$, the maps are defined as

$$\phi_i(e_{m_1 \cdots m_i}) = \sum_k (-1)^{k-1} x_{m_k} e_{m_1 \cdots \widehat{m_k} \cdots m_i}$$

  The dual of this algebra is the exterior algebra on n generators $\wedge(x_1, \ldots, x_n)$.

- The second example is inspired by algebraic geometry. Consider

$$A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_r),$$

  where $(f_1, \ldots, f_r)$ forms a regular sequence on R (see Definition 2.1.1). A ring like this is called a complete intersection. In algebraic geometry, a projective variety $V \subset \mathbb{P}^n$ is called a complete intersection if it can be defined by the vanishing of homogeneous polynomials, as many as the codimension of V in $\mathbb{P}^n$. Each of these polynomials defines a hypersurface, and V should be exactly equal to the intersection of these hypersurfaces. The reason why these varieties are studied a lot is that they show up all over

the place: one can prove that given a number of transversely intersecting hypersurfaces in $\mathbb{P}^n$, their intersection is always a complete intersection. This follows from Bertini's theorem, see Harris [42], Chapter 17. One should however note that the class is also small, because there are a whole bunch of restrictions on the topology of these intersections; an example being that complete intersections of dimension 2 or more have to be simply connected. In any case, one can prove that the coordinate ring of V satisfies the definition of a complete intersection ring. For a complete intersection of quadrics, it has been proved by Tate [76] that A is Koszul.

- For I an ideal in $k[x_1, \ldots, x_n]$ generated by an arbitrary set of monomials (always of degree two of course), $A = k[x_1, \ldots, x_n]/I$ is Koszul, see Fröberg [32]. These kind of rings naturally arise in algebraic combinatorics and combinatorial commutative algebra. Moreover, one can change the coordinate ring of any affine algebraic variety to a monomial algebra in an algorithmic way, obtaining a flat degeneration of the variety. These degenerations preserve some nice properties: the dimension, degree, and Hilbert series are all equal, and can be computed in the degeneration, using combinatorial techniques introduced by Stanley and Reisner [74]. Let us illustrate this technique on a small example: take $R = k[a, b, c, d]/(ac - b^2, bd - c^2, ad - bc)$. Then $\mathrm{Proj}(R)$ is the twisted cubic: the image of the 3-uple Veronese embedding
$$\mathbb{P}^1 \to \mathbb{P}^3 : [s : t] \mapsto [s^3, s^2t, st^2, t^3].$$
The degeneration of I is obtained by finding a Gröbner basis, and if we fix lexicographical ordering $a < b < c < d$, the generators ac, bd and ad form such a basis. Indeed, the overlaps are given by:
$$(ad)b = bbc \qquad a(db) = (ac)c = bbc$$
$$c(ad) = bcc \qquad (ca)d = b(bd) = bcc$$
The degenerate ideal is then $\tilde{I} = (ac, bd, ad)$. This is the Stanley-Reisner ideal corresponding to the

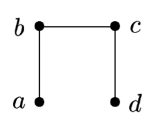


Figure 2.1: Simplicial complex - Twisted cubic

simplicial complex $\Delta = \{\emptyset, a, b, c, d, ab, bc, cd\}$ with vertices $V = \{a, b, c, d\}$, see Figure 2.1. Indeed, the definition of the Stanley-Reisner ring of an abstract simplicial complex Δ is $k[\Delta] = S/I_\Delta$, where
$$S = k[x_v : v \in V], \text{ and}$$
$$I_\Delta = \left(\prod_{j=1}^{r} x_{v_{i_j}} \mid \{v_{i_1}, \ldots, v_{i_r}\} \notin \Delta \right)$$
What happens geometrically is contained in Figure 2.2. From their work, it then follows that
$$\dim\, k[\Delta] = \dim\, \Delta + 1 = 2,$$
which seems logical since we're looking at the equations of the cone over the twisted cubic. Remember that the degree of a projective variety V embedded in $\mathbb{P}^n$ is an extrinsic property defined to be the

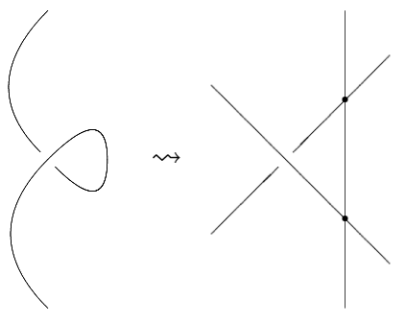


Figure 2.2: Flattening out - Twisted cubic

number of k-intersection points with a linear subspace L in general position, and $\dim(V)+\dim(L) = n$. Then,

$$\deg k[\Delta] = |\text{maximal faces of } \Delta| = 3.$$

This agrees with the fact that the projective line has an essentially unique embedding of degree n in $\mathbb{P}^n$, given by the rational normal curves. The Hilbert series also has a very nice combinatorial description (and allows for extraction of both degree and dimension), but we won't get into that. For more, consult Stanley [74].

- Taking our cue from algebraic geometry once again, suppose R is a commutative Cohen-Macaulay ring, that is, for all ideals I the equality

$$\operatorname{codim}(I) = \operatorname{depth}(I)$$

holds. The depth is calculated by looking at R as an R-module, and $\operatorname{codim}(I) = \min\{\operatorname{codim}(\mathfrak{p}) \mid I \subset \mathfrak{p},\ \mathfrak{p} \text{ prime}\}$, and the codimension of a prime ideal is the dimension of the local ring $R_{\mathfrak{p}}$. If R is a domain, finitely generated over a field, and I an ideal, then $\operatorname{codim}(I) = \dim(R) - \dim(I)$, as one would expect. Like we said before, these are rings possessing some properties of non-singular varieties, while still being sufficiently general to include a lot of non-trivial examples. For R a Noetherian graded ring, define the codimension of the ring itself to be

$$\operatorname{codim}(R) = \operatorname{edim}(R) - \dim(R),$$

where edim denotes the embedding dimension: this reduces to $\dim_k(R_1)$ if I does not contain any linear forms (as in our usual setting). Abhyankar [1] proved that for Cohen-Macaulay rings, the following inequality holds:

$$\deg(R) \geq 1 + \operatorname{codim}(R).$$

Rings for which equality holds are called rings of minimal degree or maximal codimension, and these are Koszul. An explicit example is $k[x_{ij},\ 1 \leq i \leq 2, 1 \leq j \leq n]/I$, where I is the ideal generated by the maximal minors of the matrix $(x_{ij})_{ij}$. More can be found in Schenzel [70].

- Particular examples of Cohen-Macaulay rings are Gorenstein rings. These rings are used to study dualities in algebraic geometry and occupy the important place

$$\text{Regular rings} \ \subset\ \text{Complete intersections} \ \subset\ \text{Gorenstein rings} \ \subset\ \text{Cohen-Macaulay rings.}$$

A ring A is Gorenstein if for every prime $\mathfrak{p}$, the localized ring $A_{\mathfrak{p}}$ has finite injective dimension. These are the rings for which the canonical module associated to the (usually singular) variety is free. In the non-singular, affine case, the canonical module is made up of sections of the canonical bundle, which is the top exterior power of the cotangent bundle. These sections form a module over the coordinate ring of the variety. For a good introduction to the (Grothendieck) duality theory that goes along with these rings, see Altman and Kleiman [2]. From an algebraic point of view, it has really been Hyman Bass' [10] splendid article that has opened up a plethora of investigations. We only mention the impressing number of different characterizations of Gorenstein rings that Bass put together:

Theorem 2.4.1 *(Ubiquity) Let $(R, \mathfrak{m})$ be a Noetherian local ring. The following are equivalent:*

1. *If R is the homomorphic image of a regular local ring, then R is Gorenstein in the geometric sense of Grothendieck-Serre.*
2. $\mathrm{id}_R(R) < \infty$
3. $\mathrm{id}_R(R) = \dim(R)$.
4. *R is Cohen-Macaulay and some system of parameters generates an irreducible ideal.*
5. *R is Cohen-Macaulay and every system of parameters generates an irreducible ideal.*
6. *If $0 \to R \to E^0 \to \cdots$ is a minimal injective resolution of R, then for each $h \geq 0$, $E^h \cong \sum_{\mathrm{ht}(p)=h} E_R(R/p)$.*

Just like in the previous example, there is a notion of extremality, and it turns out that also in this case, one gets Koszul algebras. For a concrete example, take a skew-symmetric 5×5-matrix, and take I to be the ideal generated by the 4×4 Pfaffians.

- Another important example comes from Veronese subalgebras. Given a graded algebra A, the dth Veronese subalgebra is $A^{(d)} = \oplus_{i \geq 0} A_{id}$. The motivation comes from the Veronese embedding, which is given by taking the Proj of the inclusion $A^{(d)} \to A$, for $A = k[x_1, \ldots, x_n]$. For two graded algebras A and B, one defines the Segre product as $A \circ B = \oplus_{i \geq 0} A_i \otimes_k B_i$. Again, the terminology comes from geometry, where the Segre embedding is a way of turning the cartesian product of projective varieties into a projective variety. Barcanescu and Manolache [9] show that starting with a polynomial ring, performing any finite number of Segre products and Veronese subalgebras will always remain Koszul. For more about this, see the next section.

- The free polynomial algebra $k\langle x_1, \ldots, x_n \rangle$ is a Koszul algebra, using the obvious resolution

$$0 \longrightarrow A^n \xrightarrow{\phi_1} A \longrightarrow k \longrightarrow 0$$

  with ϕ_1 given by $(x_1 \cdots x_n)$. Its Koszul dual is $k\langle y_1, \ldots, y_n \rangle / (y_i y_j, 1 \leq i, j \leq n)$.

- The monomial result above can be generalized to the non-commutative setting. Given an ideal I generated by an arbitrary set of monomials of degree two in $k\langle x_1, \ldots, x_n \rangle$, then $k\langle x_1, \ldots, x_n \rangle / I$ is Koszul. These results have been used by Fröberg to show that some nice varieties have rational Poincaré series. Another application is to calculate the number of walks of certain kinds in a directed graph, see the references in Fröberg [33].

- It is not necessary to work with connected algebras, where $A_0 = k$. One could also take $A_0 = k \times k \times \cdots \times k$, getting graded quiver algebras: a quotient of the path algebra kQ of a quiver by an admissible ideal I, meaning that $I \subset (kQ^+)^2$ (see Section 3.2 for a small introduction to quivers). In this setting one can prove that for example hereditary algebras, or preprojective algebras of non-Dynkin quivers are Koszul. An especially nice class of algebras falling under this category is the class of selfinjective Koszul algebras, which can be used to show that quadratic Artin-Schelter algebras of global dimension 2 are Noetherian iff they have finite GK-dimension, see the work of Martinez-Villa [41]. He is also able to obtain partial answers to a question by Hartshorne [43] on the existence of locally free sheaves of small rank on projective space, using these Koszulity results, see [57].

Applications

Projective varieties

The first application is actually a huge class of examples. In [7], Backelin proved that for any graded ring $A = k[x_1, \ldots, x_n]/I$, where I is a homogeneous ideal, a high enough Veronese subalgebra is Koszul. More precisely, for some d,

$$A^{(d)} = \oplus_{i \geq 0} A_{id}$$

is Koszul. The importance of this results stems from the simple

Proposition 2.4.2 *For A as above, and any natural number d, we have*

$$Proj(A) \cong Proj(A^{(d)}).$$

Proof: There is a bijective correspondence between homogeneous prime ideals of A and of $A^{(d)}$ by $P \mapsto P \cap A^{(d)}$. Suppose P and Q are two different homogeneous primes of A, such that $P \cap A^{(d)} = Q \cap A^{(d)}$. Take $f \in P, f \notin Q$, such that f is homogeneous (this is always possible). Then $f^d \in A^{(d)} \cap P$, and thus $f^d \in Q$. Since Q is prime this would imply $f \in Q$, which is a contradiction, so the map is injective. To prove surjectivity, we'll use the lying over property. Remember that this says that if $R \subset S$ is an integral extension, and P is a prime ideal of R, then there exists a prime ideal Q of S such that $Q \cap R = P$. It remains to prove that $A^{(d)} \subset A$ is an integral extension. It suffices to prove this for homogeneous elements since integral elements form a subring. For $a \in A$ homogeneous, it is clear that a is a zero of $f(x) = x^d - a^d$, and $a^d \in A^{(d)}$. On the level of sheafs, note that the ring of fractions $A^{(d)}[1/f^d] \cong A[1/f]$, for f homogeneous, just by multiplying numerator and denominator by a sufficient power of f. $\square$

This proposition, in conjunction with Backelin's theorem, implies that every projective variety has a coordinate ring that is Koszul. We will briefly discuss an alternative proof of Backelin's theorem due to Eisenbud, Reeves and Totaro [25]. Their statement is actually stronger, but we will first need some terminology. Put a random monomial ordering on $S = k[x_1, \ldots, x_n]$ starting from $x_1 > \cdots > x_n$, and for $p \in S$, denote by $in_>(p)$ the dominant term with respect to this ordering. For a homogeneous ideal I of S, denote

$$in_>(I) = (in_>(p) \mid p \in I),$$

the initial ideal. Denote $T_d = k[x_m \mid m \text{ monomial of degree } d \text{ in } S]$, and by $\phi : T_d \to S$, the map that sends $x_m \mapsto m$. It is then immediate that for A as before, $A^{(d)} = T_d/V_d(I)$, where $V_d(I) = \phi^{-1}(I)$. The minimal generators of I are the homogeneous elements of I not in $(x_1, \ldots, x_n)I$, and by $\delta(I)$, we denote the

maximum of the degrees of the minimal generators. By $\Delta(I)$ we denote the minimum, over all choices of variables and monomial ordering, of the maximum of the degrees of minimal generators of $in_> I$. The proof begins by showing that for sufficiently large d, $\Delta(V_d(I)) = 2$, meaning that for $V_d(I)$, there is a choice of variables and order such that $in_>(V_d(I))$ can be generated by monomials of degree 2: we say $V_d(I)$ admits a quadratic initial ideal. Furthermore, any graded algebra admitting a quadratic initial ideal is shown to be Koszul, proving Backelin's theorem. To get a feel for the necessary techniques, we prove a much easier, though related theorem mentioned by Mumford in [60], and then go over the main steps in the proof of the above theorems. Mumford's result (which seems to be a lot older) basically says that each projective variety has a quadratic coordinate ring.

Proposition 2.4.3 *For $I \subset S$ a homogeneous ideal, and d fixed,*

$$\delta(V_d(I)) \leq \max(2, \lceil \delta(I)/d \rceil).$$

Thus, for d high enough ($d \geq \delta(I)/2$), $V_d(I)$ is generated by quadrics, and the previous proposition then implies Mumford's result.

Proof: First notice that $V_d(I)$ is generated by

- the kernel of ϕ
- $\phi^{-1}(\{\text{elements of degree } kd \text{ in } (x_1, \ldots, x_n)^{kd-e} g\})$, where kd is the smallest k-multiple that is $\geq e$, and g is a degree e generator of I

If we were to know that $\mathrm{Ker}(\phi)$ is generated by quadratic forms, then this would imply the proposition. To prove this, we use some basic algebraic geometry. The monomials of degree d in $k[x_0, \ldots, x_n]$ form a subspace of dimension $\binom{n+d}{d}$. Denote $n_d = \binom{n+d}{d} - 1$; then the Veronese map is defined by

$$v : \mathbb{P}^n \to \mathbb{P}^{n_d} : (x_0 : \ldots : x_n) \mapsto (x_0^d : x_0^{d-1} x_1 : \ldots : x_n^d),$$

the image running over all monomials of degree d, written in lexicographical order. The defining polynomials all have the same degree, and do not all simultaneously vanish, so this is a well defined map of projective varieties. This map is actually an embedding of projective varieties. To prove this, we define the inverse map. Let $W \subset \mathbb{P}^{n_d}$ be the image of v, and denote homogeneous coordinates on $\mathbb{P}^{n_d}$ by z_I, $I = (i_0, \ldots, i_n) \in \mathbb{N}^{n+1}$ running over multi-indices such that $\sum i_k = d$. At each point of W, at least one of the coordinates indexed by monomials $x_0^d, \ldots, x_n^d$ is non-zero, and denote by $U_i \subset W$ the subset where the coordinate matching x_i^d is non-zero. These sets cover W, and by defining

$$U_i \to \mathbb{P}^n : z \mapsto [z_{(1,0,\ldots,d-1,\ldots,0)} : \ldots : z_{(0:\ldots,d-1,0,\ldots,1)}],$$

where the $d-1$ appears in the i-th place, we get a map $W \to \mathbb{P}^n$, since they agree on all overlaps $U_i \cap U_j$, which is inverse to the Veronese map. The image of the Veronese map is easily checked to be the projective variety defined by the polynomials

$$z_I z_J - z_K z_L, \ I, J, K, L \in \mathbb{N}^{n=1}, \ I + J = K + L,$$

and thus quadratic, proving the proposition. □

To formulate the first important theorem, we need the notion of Castelnuovo-Mumford regularity of I, which is a stable measure of the size of the generators of I, see Chapter 20 in Eisenbud [24] for a detailed treatment.

Definition 2.4.4 *For $I \subset S$, the (Castelnuovo-Mumford) regularity of I is defined as*

$$reg(I) = \max\{t_i^S(I) - i \mid i \geq 0\},$$

where $t_i^S(I) = \max\{j \mid \mathrm{Tor}_i^S(k, I)_j \neq 0\}$, and $\mathrm{Tor}_i^S(k, I)_j$ denotes the j-th graded piece.

By looking at $i = 0$, we see that the regularity is $\geq$ the maximal degree of the generators of I. The last definition we need is

Definition 2.4.5 *For a monomial $m \in S$, write $\max(m)$ for the largest index i such that $x_i|m$. A monomial ideal I is called combinatorially stable if for every monomial $m \in I$, and $j < \max(m)$, the monomial $(x_j/x_{max(m)})m \in I$.*

Theorem 2.4.6

$$\Delta(V_d(I)) \leq \max(2, \lceil \mathrm{reg}(I)/d \rceil)$$

In particular, if $d \geq \mathrm{reg}(I)/2$, then $\Delta(V_d(I)) = 2$.

Proof:[sketch] If I is an ideal in generic coordinates, and $\mathrm{reg}(I) = e$, then $in(I)$ is generated in degrees $\leq e$, using reverse lexicographical order. This follows from an alternative characterization of regularity by Bayer and Stillman, see Theorem 9 in [25]. The same theorem also implies that $in(I)$ includes all monomials in $x_1, \ldots, x_{n-j}$ (for some fixed j) of degree e, and for every monomial $m \in in(I)$ of degree e with $\max(m) > n-j$, $in(I)$ also contains $x_k/x_{\max(m)} \cdot m$, $1 \leq k \leq \max(m)$. These two facts taken together imply that $in(I_e)$ is combinatorially stable with respect to reverse lexicographical order. If we were to know that for all ideals $J \subset S$, such that $in(J)$ is combinatorially stable, it would hold that

$$\delta(in(V_d(J))) \leq \max(2, \lceil \mathrm{reg}(J)/d \rceil),$$

then we would be done since

$$\Delta(V_d(I)) \leq \delta(in_{>'}(V_d(gI))) \leq \max(2, \lceil \mathrm{reg}(gI)/d \rceil) = \max(2, \lceil \mathrm{reg}(I)/d \rceil),$$

for a general choice of coordinates g, and $>'$ the induced order on T_d. This is exactly what they prove, the argument essentially being based on a technical proposition that says that $in(ker(\phi))$ is generated by quadratic forms for every d. $\square$

With this theorem established, it remains to prove that a graded algebra admitting a quadratic initial ideal is Koszul. To do this, we define the rate of A, a measure of the rate of growth of the degrees of the syzygies in a minimal free resolution.

Definition 2.4.7 *The rate of A is defined to be*

$$rate(A) = \sup\left\{ \frac{t_i^A(k) - 1}{i - 1} \mid i \geq 2 \right\}.$$

In particular, A is Koszul iff rate(A)=1.

The theorem giving the desired result then says that $\mathrm{rate}(A) \leq \Delta(I) - 1$. In particular, if $\Delta(I) = 2$, then A is Koszul. The authors prove the theorem by framing it in the more general context of multihomogeneous modules and then specializing to the case at hand, see [25], Section 4.

Coherent sheaves on projective space

Fix a vector space $V = kx_0 + \ldots + kx_r$, with basis elements of degree 1; the dual basis elements $e_0, \ldots, e_r$ are given degree -1. Skew-commutativity of the exterior algebra $E = \wedge^\bullet V^*$ ensures that it behaves a lot like a commutative local ring. Any one-sided graded ideal is automatically two-sided, any finitely generated graded E-module P has unique minimal free graded resolution $\mathbf{F}$, and $\mathrm{Tor}^E(P, k) = \mathbf{F} \otimes_E k$ as graded vector spaces. In fact, in this case the construction even works for non finitely generated modules, since E has a unique maximal ideal (V^*), and Nakayama still holds. Any graded left E-module P can be regarded as graded right E-module, though signs tend to blur things: for $p \in P$ and $e \in E$ homogeneous elements, we have $pe = (-1)^{\deg(p)\deg(e)}ep$. From now on we assume all modules are graded left modules. If $P = \oplus P_i$ is a finitely generated E-module, then the vector space dual $P^* = \oplus P_i^*$ is naturally a right E-module, where $(\phi \cdot e)(p) = \phi(ep), \phi \in P_i^*, e \in E_{-j}$ and $p \in P_{i+j}$. As graded left module, with $(P^*)_{-i} = P_i^*$, we have

$$(e\phi)(p) = (-1)^{\deg(e)\deg(\phi)}(\phi e)(p) = (-1)^{\deg(e)\deg(\phi)}\phi(ep).$$

Denote $S = k[x_0, \ldots, x_r]$. Then $S \otimes P$ becomes a linear complex of graded free S-modules,

$$\mathbf{L}(P) : \cdots \to S \otimes P_i \to S \otimes P_{i-1} \to \cdots,$$

sending $1 \otimes p \mapsto \sum x_i \otimes e_i p$, and $S \otimes P_i \cong S(-i)^{\dim(P_i)}$ is in homological degree i, and generated in degree i. This is a complex because

$$d_{i-1}d_i(p) = \sum_j \sum_i x_j x_i \otimes e_j e_i p = \sum_{i \leq j} x_j x_i \otimes (e_j e_i + e_i e_j)p = 0.$$

In matrix terms, if $\{p_s\}_s, \{p'_t\}_t$ are bases for P_i and P_{i-1}, and

$$e_m p_s = \sum_t c_{m,s,t} p'_t,$$

then the matrix of d_i is of the form

$$(d_i)_{t,s} = \sum_m c_{m,s,t} x_m.$$

In fact, $\mathbf{L}$ is a functor from the category of graded E-modules to the category of linear free complexes of S-modules, and we have

Proposition 2.4.8 *The functor* $\mathbf{L}$ *defines an equivalence from the category of graded* E*-modules to the category of linear free complexes of* S*-modules.*

Proof: For each $e \in V^*$, and any vector space P, there is a unique linear map $e : V \otimes P \to P$ satisfying $e(x \otimes p) = e(x)p$. Given a linear free complex of S-modules

$$\cdots \to S \otimes P_i \to S \otimes P_{i-1} \to \cdots,$$

we have that $d(P_i) \subset V \otimes P_{i-1}$, so one can define a multiplication

$$V^* \otimes P_i \to P_{i-1} : e \otimes p \mapsto e(d(p)).$$

An unpleasant computation shows that the anti-commutative and associative laws hold, and that this construction defines an inverse to $\mathbf{L}$. $\square$

Now any finitely generated graded S-module can be sheafified to a coherent sheaf on the projective space $\mathbb{P}(V)$, and thus, the above complex becomes a complex of coherent sheaves. This construction then gives a correspondence between graded modules over $\wedge^\bullet V^*$ and bounded complexes of coherent sheaves on $\mathbb{P}(V)$. The exact form of this correspondence is due to Bernstein-Gel'fand-Gel'fand [14], and states that there is an equivalence of triangulated categories between the bounded derived category of coherent sheaves on $\mathbb{P}(V)$ and the *stable* module category of finitely generated graded left $\wedge^\bullet V^*$-modules. This is only the beginning of a grand unification, put forward in a paper by Beilinson, Ginzburg and Soergel [11] where they prove the following

Theorem 2.4.9 *Let A be a Koszul ring (where this time A_0 can be any semisimple ring). Under the assumption that A is a finitely generated A_0-module from the left and from the right, and $A^!$ is left Noetherian, there is an exact functor*

$$K : D^b(A\text{-}\underline{\mathrm{mod}}) \to D^b(A^!\text{-}\underline{\mathrm{mod}}),$$

that is an equivalence of triangulated categories, where $\underline{\mathrm{mod}}$ *denotes the category of finitely generated graded modules.*

Remark 2.4.10 *There is an even vaster framework describing Koszul duality, which relates it to operad theory and graph cohomology, though we did not dare venture too far into these waters. The Ishmaels out there can consult Ginzburg and Kapranov [39].*

To end, we briefly discuss some more recent work about the classic BGG correspondende by Eisenbud, Fløystad and Schreyer [26, 30]. The algebra E is projective and injective as module over itself: injectivity follows from the well known result that Frobenius algebras are self-injective, and the Frobenius form on E is given by taking the coefficient of the grade $r+1$ part of the product of two elements (remember V has dimension $r+1$). If P is a finitely generated graded E-module then splicing together a minimal projective and a minimal injective resolution, we get an acyclic complex called the Tate resolution T of P. If P corresponds to a coherent sheaf $\mathcal{F}$ under the BGG correspondence, then all of the sheaf cohomology groups of all twists of $\mathcal{F}$ can be read off of this Tate resolution. Now it turns out that T is completely determined by an arbitrary differential $T^i \to T^{i+1}$. This sets up a correspondence between homogeneous matrices of exterior forms with no non-zero constant entries and complexes of coherent sheaves on $\mathbb{P}(V)$. The authors are able to use these techniques to simplify some beautiful geometric constructions. The Horrocks-Mumford bundle is the only known indecomposable vector bundle of rank 2 on $\mathbb{P}^4$, and its geometry is related to the icosahedron and modular surfaces, see Hulek [44]. For a vector space $V^* = ke_1 + ke_2 + ke_3 + ke_4 + ke_5$, define the matrix

$$A = \begin{pmatrix} e_1 \wedge e_2 & e_2 \wedge e_3 & e_3 \wedge e_4 & e_4 \wedge e_5 & e_5 \wedge e_1 \\ e_3 \wedge e_5 & e_4 \wedge e_1 & e_5 \wedge e_2 & e_1 \wedge e_3 & e_2 \wedge e_4 \end{pmatrix}$$

This gives a map $E(-2)^5 \to E^2$, and via the BGG-correspondence, Fløystad shows this returns the Horrocks-Mumford bundle.

Representations of Sklyanin algebras

Koszul duality for non-commutative algebras can be used to learn a great deal about their representations. Sklyanin algebras are non-commutative deformations of polynomial rings, in the sense that they are generated in degree 1, have Hilbert series $(1-t)^{-n}$, are left and right Noetherian domains of global dimension

n, are Auslander-Gorenstein, Cohen-Macaulay, and in fact Koszul. The previous properties allow one to associate actual (algebraic) geometric objects to them. In fact, in global dimension 3, these algebras belong to a family that is completely classified by geometric data: a scheme (either $\mathbb{P}^2$ or a cubic divisor on $\mathbb{P}^2$) and an automorphism on it. For a systematic study of these algebras and their geometry, one can consult Artin-Tate-Van den Bergh [4], Le Bruyn [52, 53], Smith [72] and others; we will only briefly discuss how Koszulity comes into play. For each $n \geq 3$, an elliptic curve E over an algebraically closed field k, and a translation automorphism σ of this curve, one has a Sklyanin algebra $A = A_n(E, \sigma)$. The general representation theory of these algebras is not well-understood for high n, but a particular class of graded modules, called linear modules, can be understood using Koszul duality. In the language we used in this chapter, the importance of linear modules stems from the fact that they have linear resolutions, and $\mathrm{Ext}_A(-, k)$ is a duality between the category of graded modules over A and $A^!$ having linear resolutions, so we can understand the linear modules of A via $A^!$, which is a finite dimensional algebra.

To give at least a clue of what a Sklyanin algebra is, we'll consider the $n = 3$ case. Then there exist scalars $a, b, c \in k$ depending on E and σ such that $A_3(E, \sigma) = TV/(R)$, for $V = kx + ky + kz$, and

$$(R) = \begin{pmatrix} ax^2 + byz + czy \\ ay^2 + bzx + cxz \\ az^2 + bxy + cyx \end{pmatrix}$$

Obviously $(a : b : c) \in \mathbb{P}^2$, though not all triples can be used: some non-degeneracy condition has to be satisfied. In this case $E \subset \mathbb{P}^2$ is an elliptic curve, V consists of the global sections of some line bundle $\mathcal{L}$ of degree 3 on E, and the space of relations R consists of those sections $f \in V \otimes V$ that vanish on the shifted diagonal $\Delta_\sigma = \{(p, \sigma(p)) \mid p \in E\}$. With the definition above we think of $V = A_1$ as linear forms on $\mathbb{P}^2 = \mathbb{P}(V^*)$ and of $V \otimes V$ as bilinear forms on $\mathbb{P}^2 \times \mathbb{P}^2$. In geometric language, R can thus be viewed as $H^0(E \times E, (\mathcal{L} \boxtimes \mathcal{L})(-\Delta_\sigma))$, where $\boxtimes$ denotes $pr_1^* \otimes pr_2^*$ as usual.

Definition 2.4.11 *A finitely generated graded A-module M is called d-linear if it is cyclic and the Hilbert series of M is the same as that of a linear subspace of projective space: $h_M(z) = 1/(1-z)^d$.*

Linear modules correspond to linear subspaces of $\mathbb{P}(A_1^*)$: cyclicity means there is a surjective map $A \to M$ and thus, a surjective map $A_1 \to M_1$. Because of the Hilbert series, the kernel of this map is of dimension $m - d$, with $m = \dim(A_1)$, so $\dim(\mathrm{Ker}^\perp) = d$ which corresponds, upon projectivizing, to a linear subspace of dimension $d-1$ in $\mathbb{P}(A_1^*)$. Again, in the simplest case $n = 3$, point modules ($d = 1$) correspond bijectively to points of E, and line modules ($d = 2$) correspond bijectively to lines in $\mathbb{P}^2$ (if σ is not of order 3). For more about linear modules on Sklyanin algebras and their importance, we refer to Smith [73]. One can show that for Sklyanins, linear modules have linear resolutions, so having a duality between A-modules with linear resolutions and $A^!$-modules with linear resolutions would imply we could study the class of linear modules of A by studying certain modules over its Koszul dual. Now we already know that for Koszul algebras $A^! = \mathrm{Ext}_A(k, k)$, and in fact there is a functor

$$\mathrm{Ext}_A(-, k) : \mathrm{Mod}(A) \to \mathrm{Mod}(A^!) : M \mapsto M^\dagger$$

Using many of the same techniques we used in this chapter, one then shows that for a module with linear resolution L, its image $L^\dagger$ also has a linear resolution, and $L^{\dagger\dagger} \cong L$.

Remark 2.4.12 *There is significant overlap between this application and the previous one, though the exact connection is not completely trivial. The correspondence can be found in Section 7 of [73].*

3

N-Koszul algebras and distributivity

Given the enormous amount of examples and applications of Koszulity, a generalization of the concept has been introduced to take algebras with cubic, quartic, ... relations along for the ride. In non-commutative geometry, the motivating examples come from AS-regular algebras, say of dimension 3. It has been shown by Artin and Schelter [3] that these algebras have either 3 generators and 3 quadratic relations, or 2 generators and 2 cubic relations. The first kind is Koszul, though the second obviously can't be. It has however been shown that they are '3-Koszul'. The notion of distributivity brings a combinatorial description of Koszulity to the table, and allows for a whole new approach. The main point however is that the definition of distributivity works for relations of arbitrary degree, and seems to unify all the N-Koszul algebras.

3.1 Lattices and distributive algebras

Definition 3.1.1 *An algebra of the form $A = TV/(R)$, where $R \subset V^{\otimes N}$ is a subspace of the finite dimensional k-vector space V, is called N-Koszul if the free modules occurring in a minimal resolution of k are generated in degrees $0, 1, N-1, N, 2N-1, 2N, \ldots$*

One of the reasons distributivity has been studied in Kriegk and Van den Bergh [51], is that it seems to capture the previous definition, independent of N. More precisely, there is a promising theorem by Berger [12] which says

$$\text{Distributivity} \ + \ \text{Extra condition} \ \Rightarrow \ N\text{-Koszul} \ \Rightarrow \ \text{Extra condition}.$$

This extra condition will be discussed thoroughly a little further down. It is unknown whether the first implication is reversible, or if counterexamples exist. In a way which we will make precise below, distributive algebras can be handled set-theoretically, greatly simplifying their study. This turns out to be crucial in determining the representations of certain algebras associated to them, see Chapter 4. Before defining distributive algebras, we review a bit of lattice theory and focus in particular on properties of lattices of vector spaces. Remember that a lattice is given by a discrete set Ω, endowed with two binary, idempotent operations $\wedge$ and $\vee$, respectively called 'meet' and 'join', which are commutative, associative, and satisfy the absorption identities

$$a \wedge (a \vee b) = a, \ (a \wedge b) \vee b = b.$$

Further, the lattice is called distributive if the following identity holds:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

A weaker condition is modularity, where one only requires the distributive identity for elements a, b and c satisfying the equivalent conditions

$$c \wedge a = c, \text{ or } c \vee a = a.$$

It can easily be shown that the distributive identity is equivalent to the following one, and both do not change after permuting a, b and c:

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

A sublattice of a lattice Ω is a subset closed under meet and join. Given a subset $X \subset \Omega$, the lattice generated by X consists of all elements of Ω that can be obtained from elements of X by meets and joins. Note that a finitely generated distributive lattice is finite. In practice, it can be hard to check if the lattice generated by some set X is distributive, since this could amount to checking an infinite number of equations. One would like to have a finite system of equations in the elements of X that guarantees distributivity of the lattice generated by X.

Theorem 3.1.2 *A subset $x_1, \ldots, x_N \in \Omega$ of a modular lattice Ω generates a distributive sublattice if and only if for any sequence of indices $1 \leq i_1 < \cdots < i_l \leq N$ and any number $2 \leq k \leq l-1$ the triple*

$$x_{i_1} \vee \cdots \vee x_{i_{k-1}}, x_{i_k}, x_{i_{k+1}} \wedge \cdots \wedge x_{i_l}$$

satisfies the distributivity equations.

Proof: See Theorem 6.3 in Polishchuk and Positselski [66]. □

From now on we put Ω equal to the set of all linear subspaces of a given vector space W. The operations $\wedge$ and $\vee$ are defined by

$$X \wedge Y = X \cap Y, \; X \vee Y = X + Y.$$

It is easy to see that this lattice is not distributive, only modular: take three different lines through the origin in $\mathbb{R}^2$. The next proposition is crucial to all that follows.

Proposition 3.1.3 *Given a vector space W, and a collection of subspaces $X_1, \ldots, X_N$, the following are equivalent:*

1. *the collection $X_1, \ldots, X_N$ generates a distributive sublattice.*
2. *there exists a direct sum decomposition $W = \oplus_\eta W_\eta$ such that each of the subspaces X_i is the sum of a set of subspaces W_η.*
3. *there exists a basis $(w_\alpha)_\alpha$ of W such that each of the subspaces X_i is in the linear span of some of these w_α.*

Proof: It is obvious that (2) $\Rightarrow$ (1) and (2) $\Leftrightarrow$ (3); we prove (1) $\Rightarrow$ (2). For some subset $\eta \subset \{1, \ldots, N\}$, choose a subspace W_η in $\cup_{i \in \eta} X_i$ complementary to the subspace

$$(\cap_{i \in \eta} X_i) \cap (\sum_{j \notin \eta} X_j).$$

The proof would be complete if we show that the subspaces W_η

- are linearly independent
- generate W
- $X_i = \sum_{\eta \ni i} W_\eta$

We'll start with the last condition: more generally we prove by descending induction that

$$\sum_{\eta \supset \gamma} W_\eta = \cap_{i \in \gamma} X_i,$$

for some subset $\gamma \subset \{1, \ldots, N\}$. For $\gamma = \{1, \ldots, N\}$, this is clear since

$$W_{\{1 \ldots, N\}} = X_1 \cap \cdots \cap X_N.$$

Suppose this is true for all strictly larger sets $\gamma' \supset \gamma$. It suffices to apply distributivity to the definition of W_γ to see that this equation holds. Looking at it for the empty set, we see that the W_η generate W. Remains to show the linear independence. Suppose $\sum_s w_{\eta_s} = 0$, for nonzero $w_{\eta_s} \in W_{\eta_s}$, and the η_s are distinct subsets. Pick an s_0 such that η_{s_0} does not contain any other of these subsets. Then,

$$\sum_{s \neq s_0} W_{\eta_s} \subset \sum_{s \neq s_0} \cap_{j \in \eta_s} X_j \subset \sum_{j \notin \eta_{s_0}} X_j,$$

which is impossible since by definition of the W_η, W_{s_0} does not intersect $\sum_{j \notin \eta_{s_0}} X_j$. □

Using this proposition we can now give and illustrate the definition of a distributive algebra [51]:

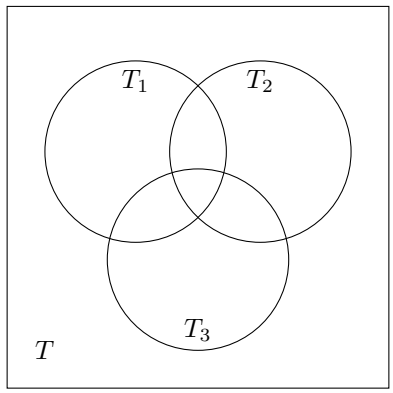
Definition 3.1.4 *An algebra A of the form $TV/(R)$, where V is a finite dimensional k-vector space, and $R \subset V^{\otimes N}$ is a linear subspace, is called distributive if in each degree, the subspaces of relations generate a distributive lattice. More formally, for every natural number n, the subspaces*

$$R \otimes V^{\otimes n-N}, V \otimes R \otimes V^{n-N-1}, \ldots, V^{\otimes i} \otimes R \otimes V^{\otimes n-N-i}, \ldots, V^{\otimes n-N} \otimes R \subset V^{\otimes n}$$

generate a distributive lattice.

The motivation underlying this definition comes from a theorem by Backelin [6] which says that in the case $N = 2$, distributivity is the same as Koszulity, giving us many examples and a completely new way of checking whether a given algebra is Koszul. Since the concept of distributivity makes sense for any N, this can be viewed as another possible way of defining higher Koszul algebras. Using Proposition 3.1.3, distributivity allows for a change of perspective from a linear algebraic level to a set theoretic one. More precisely, one way of studying a finitely generated, homogeneous algebra A is by examining the spaces of relations in each degree. Distributivity tells us that for each degree n, there exists a basis $T = \{t_\alpha\}_\alpha$ for $V^{\otimes n}$ such that every space of relations $R_i^{(n)} = V^{\otimes i} \otimes R \otimes V^{\otimes n-N-i}$ has a basis $T_i \subset T$, for $i \in \{0, \ldots, n-N\}$. Thus, we can consider $R_i^{(n)}$ as kT_i and faithfully represent relations in degree n by Venn diagrams. In the

case $n = 3, N = 2$ for example, we get



This naive visualisation often greatly simplifies computations (as we will see explicitly in Chapter 4).

The rest of this section is directed towards proving Backelin's theorem. In a subsequent section, we will show how the theorem can be put in a more abstract framework. The classical link between distributivity and Koszulity comes from the following property of lattices of vector spaces:

Proposition 3.1.5 *Let W be a vector space with subspaces $X_1, \ldots, X_N$. Suppose that any subset of the form $X_1, \ldots, \hat{X_k}, \ldots, X_N$ is distributive. Then the following are equivalent:*

1. *the collection $X_1, \ldots, X_N$ is distributive*
2. *the following complex of vector spaces $K_\bullet(W; X_1, \ldots, X_N)$ is exact:*

$$0 \to X_1 \cap \cdots \cap X_N \to X_2 \cap \cdots \cap X_N \to X_3 \cap \cdots \cap X_N / X_1 \to$$

$$\cdots \to \cap_{s=i+1}^{N} X_s / \sum_{t=1}^{i-1} X_t \to \cdots \to$$

$$X_N/(X_1 + \cdots + X_{N-2}) \to W/(X_1 + \cdots + X_{N-1}) \to W/(X_1 + \cdots + X_N) \to 0,$$

   where Y/Z is shorthand for $Y/(Y \cap Z)$

3. *the following complex of vector spaces $B_\bullet(W; X_1, \ldots, X_N)$ is exact everywhere except at the leftmost term:*

$$W \to \oplus_t W/X_t \to \cdots \to \oplus_{t_1 < \cdots < t_{N-i}} W / \sum_{s=1}^{N-i} X_{t_s} \to \cdots \to W / \sum_s X_s \to 0$$

Proof: To prove (1) $\Leftrightarrow$ (2), note that the equations of Theorem 3.1.2 in the case of vector spaces are given by

$$(X_{i_1} + \cdots + X_{i_{k-1}}) \cap (X_{i_k} + (X_{i_{k+1}} \cap \cdots \cap X_{i_l})) = ((X_{i_1} + \cdots + X_{i_{k-1}}) \cap X_{i_k}) + ((X_{i_1} + \cdots + X_{i_{k-1}}) \cap (X_{i_{k+1}} \cap \cdots \cap X_{i_l}))$$

and

$$(X_{i_1} + \cdots + X_{i_{k-1}}) + (X_{i_k} \cap (X_{i_{k+1}} \cap \cdots \cap X_{i_l})) = ((X_{i_1} + \cdots + X_{i_{k-1}}) + X_{i_k}) \cap ((X_{i_1} + \cdots + X_{i_{k-1}}) + (X_{i_{k+1}} \cap \cdots \cap X_{i_l})),$$

and the permuted versions, for appropriate indices i_j. These are exactly the equations one needs for the complex $K_\bullet(W; X_1, \ldots, X_N)$ to be exact. Remains to prove (1) $\Leftrightarrow$ (3). Define the exact sequence of complexes

$$0 \to B_\bullet(W/X_1; (X_2 + X_1)/X_1, \ldots, (X_N + X_1)/X_1) \to B_\bullet(W; X_1, \ldots, X_N) \to B_\bullet(W; X_2, \ldots, X_N)[-1] \to 0.$$

Given that any proper subset is distributive, the third complex is exact in degree $\neq N$, and

$$H_N B_\bullet(W; X_1, \ldots, X_N) = X_1 \cap \cdots \cap X_N.$$

It follows that the second complex is exact everywhere except at the leftmost term iff the first complex is exact in degree $\neq N - 1$ and the connecting morphism

$$X_2 \cap \cdots \cap X_N \to (X_2 + X_1) \cap \cdots \cap (X_N + X_1)/X_1$$

is surjective. Now by using induction on the number of subspaces N, one sees that the distributivity of $X_1 + X_2, \ldots, X_1 + X_N$ is equivalent to the exactness of the first complex. An easy corollary of Theorem 3.1.2 (see Corollary 6.5 in [66]) is that the set $X_1, X_2, \ldots, X_N$ is distributive if and only if the sets $X_2, \ldots, X_N$ and $X_1 + X_2, \ldots, X_1 + X_N$ are distributive and the equation

$$X_1 + (\cap_i X_i) = \cap_i (X_1 + X_i)$$

holds, which is exactly what we need to conclude the proposition. □

The proof of Backelin's theorem now comes as a corollary.

Theorem 3.1.6 (Backelin's theorem) *A quadratic algebra A is Koszul if and only if it is distributive.*

Proof: Using the above proposition, it suffices to check exactness (except at the leftmost term) of the complex $B_\bullet(V^{\otimes n}; R \otimes V^{\otimes n-2}, V \otimes R \otimes V^{\otimes n-3}, \ldots, V^{\otimes n-2} \otimes R)$ for every $n \geq 0$ or of the complex $K_\bullet(V^{\otimes n}; R \otimes V^{\otimes n-2}, V \otimes R \otimes V^{\otimes n-3}, \ldots, V^{\otimes n-2} \otimes R)$. These complexes are immediately seen to coincide with the degree n part of the $B_\bullet$ or Koszul complexes of A, and the theorem follows from the characterizations of Koszulity in Theorem 2.3.19. □

3.2 The category Cube$_\bullet$

In this subsection we are going to look at Koszul algebras as certain kinds of monoidal functors from a category of quiver representations to Vect. This reformulation will allow us to fully determine the representations of the quantum endomorphism groups that will be introduced in the following chapter. We start by reviewing some basic concepts of quivers and quiver representations.

Definition 3.2.1 *A quiver is a 4-tuple $Q = (Q_0, Q_1, t, h)$ where*

- *Q_0 is a finite set of vertices*
- *Q_1 is a finite set of arrows between these vertices*
- *$t, h : Q_1 \to Q_0$ are functions that send an arrow to its tail vertex, respectively head vertex*

In less formal language, a quiver is just a finite, directed graph. Closely connected with this is the notion of quiver representation.

Definition 3.2.2 *Let Q be a quiver. A representation $V = (\{V_i\}, \{\phi_a\})$ of Q is*

- *a choice of vector space V_i for each $i \in Q_0$*
- *a choice of linear map $\phi_a : V_{t(a)} \to V_{h(a)}$ for each $a \in Q_1$*

To get a category of representations of a fixed quiver we also need morphisms.

Definition 3.2.3 *Given a quiver Q and two representations $V = (\{V_i\}, \{\phi_a\})$ and $W = (\{W_i\}, \{\psi_a\})$, a morphism $\Gamma : V \to W$ consists of a linear map $\Gamma_i : V_i \to W_i$ for each vertex, such that for each arrow $a : i \to j$, the following diagram commutes*

$$\begin{tikzcd} V_i \arrow[r, "\Gamma_i"] \arrow[d, "\phi_a"'] & W_i \arrow[d, "\psi_a"] \\ V_j \arrow[r, "\Gamma_j"'] & W_j \end{tikzcd}$$

The resulting category has direct sums, kernels and cokernels and is in fact an abelian category. This will be clear from the description as a category of modules over an algebra associated to Q. For a quiver Q, a path is a sequence of arrows such that the tail of an arrow in the sequence coincides with the head of the previous arrow, where we also consider trivial arrows e_i for i a vertex of Q (which correspond to the stationary or trivial paths). This allows for

Definition 3.2.4 *Given a quiver Q and a field k. The path algebra kQ of Q is defined to be the associative k-algebra with k-basis the set of paths, and multiplication given by concatenation if the head and tail match up, and 0 otherwise.*

With this definition the path algebra is finite dimensional if and only if there are no loops or oriented cycles (we say Q is acyclic), and the unit element is given by $\sum_{i \in Q_0} e_i$. It can then be proved that there is an equivalence of categories between the representations of Q and the (say left) kQ-modules, see Theorem 1.5 in [5]. For acyclic quivers there is an easy description of the simple and the indecomposable projective modules. Notice that the e_i are pairwise orthogonal idempotents, providing a decomposition

$$kQ = \oplus_{i \in Q_0} kQe_i.$$

The claim is that each kQe_i is indecomposable. This can (for example) be seen by calculating the endomorphism ring of kQe_i and applying Fitting's lemma which says that a module for a finite dimensional algebra is indecomposable if and only if every endomorphism is either invertible or nilpotent. By the Krull-Schmidt theorem the set of kQe_i, $i \in Q_0$ is a complete set of indecomposable projectives for kQ. Now for the simples; as a vector space, there is a decomposition

$$kQe_i = \oplus_j e_j kQe_i.$$

It is clear that $\oplus_{j \neq i} e_j kQe_i$ is a submodule of codimension 1, since the quiver has no oriented cycles, making it maximal. From general theory, one knows that the radical of an indecomposable projective is maximal,

and by definition the radical is the intersection of all maximal submodules, implying that an indecomposable projective has a unique maximal submodule which is equal to the radical. This allows us to conclude that $\mathrm{rad}(kQe_i) = \oplus_{j \neq i} e_j kQe_i$ and

$$S_i = kQe_i/\mathrm{rad}(kQe_i)$$

is just the head of kQe_i. As a representation of the quiver, this corresponds to k at the vertex i and zeroes everywhere else. It is then also clear that for $i \neq j$, S_i is not isomorphic to S_j.

Remark 3.2.5 *The beauty of the theory is amongst others its scope: every hereditary algebra (an algebra of global dimension ≤ 1) is Morita equivalent to the path algebra of some quiver. A general finite dimensional algebra can always be written (again up to Morita equivalence) as the quotient of a path algebra by an ideal of relations. Formulated another way, this says that a finite dimensional algebra is the path algebra of a 'quiver with relations' (Q, ρ), where ρ is some finite subset of paths. For this and more material on quivers, see Pierce [65], Auslander, Reiten and Smalø [5] or Gabriel and Roiter [36].*

Using these elementary facts from the representation theory of finite dimensional algebras, we can describe the category Cube_n. For fixed n, we take Q_n to be the quiver with relations that represents an n-dimensional commutative diagram. Concretely, Q_n has a vertex x_I for every $I \subset \{1, \ldots, n\}$, and an arrow $x_{IJ} : x_I \to x_J$, for all $I \subset J$. The relations are given by $x_{JK} x_{IJ} = x_{IK}$, for $I \subset J \subset K$, using functional notation. For non-positive n, we set Q_n to be the one-point quiver $x_\emptyset$, without any arrows. A representation of this quiver with relations is of course given by a representation of the quiver such that the maps respect the relations.

Definition 3.2.6 *For a given n, the category Cube_n is the category of all k-representations of Q_n. The category Cube_0 is just the category of k-vector spaces. Concretely, an object of Cube_n is given by a collection of k-vector spaces $(V_I)_{I \subset \{1,\ldots,n\}}$, and linear maps $\phi_{I,J} : V_I \to V_J$ for every subset $I \subset J$. Given three sets $I \subset J \subset K$, these maps have to satisfy $\phi_{J,K} \circ \phi_{I,J} = \phi_{I,K}$. By cube_n, we denote the full subcategory of finite dimensional representations.*

Example 3.2.7 *Let us illustrate that these categories are very nice to work with, by looking at some low-dimensional examples. For $n = 1$, the corresponding quiver is just the Dynkin quiver A_2:*

$$A_2 : \bigcirc \xrightarrow{a} \bigcirc$$

It is readily verified that the path algebra is isomorphic to the 2×2 upper triangular matrices with entries in k. Some (and in this case in fact all) finite dimensional indecomposable modules are given by:

$$S_0 : k \longrightarrow 0 \qquad S_1 : 0 \longrightarrow k \qquad P_0 : k \longrightarrow k$$

A projective resolution of (say) S_0 is now easily constructed, almost solely by reasoning pictorially:

$$0 \longrightarrow 0 \longrightarrow k \longrightarrow k \longrightarrow k \longrightarrow k \longrightarrow 0 \longrightarrow 0$$

Example 3.2.8 *For $n = 2$, the quiver looks like:*

$$\begin{tikzcd} \bullet \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & \bullet \arrow[d, "\beta"] \\ \bullet \arrow[r, "\delta"'] & \bullet \end{tikzcd}$$

This is just a commutative square: the relation is given by $\beta\alpha = \delta\gamma$. A projective resolution of $S_\emptyset$ is:

$$0 \longrightarrow \begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & k \end{tikzcd} \longrightarrow \begin{tikzcd} 0 \arrow[r] \arrow[d] & k \arrow[d] \\ 0 \arrow[r] & k \end{tikzcd} \oplus \begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ k \arrow[r] & k \end{tikzcd} \longrightarrow \begin{tikzcd} k \arrow[r] \arrow[d] & k \arrow[d] \\ k \arrow[r] & k \end{tikzcd} \longrightarrow \begin{tikzcd} k \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd} \longrightarrow 0$$

and the path algebra is 9-dimensional this time. By the algebra map:

$$\begin{aligned} \alpha &\mapsto a \otimes e_1 \\ \beta &\mapsto e_2 \otimes a \\ \gamma &\mapsto e_1 \otimes a \\ \delta &\mapsto a \otimes e_2 \end{aligned}$$

we see that the path algebra is isomorphic to $kA_2 \otimes kA_2$.

There is an external product on the Cube-categories:

$$\boxtimes : \mathrm{Cube}_m \times \mathrm{Cube}_n \to \mathrm{Cube}_{m+n} : (U_\bullet, V_\bullet) \mapsto W_\bullet,$$

such that $W_{I \cup (J+m)} = U_I \otimes V_J$, for $I \subset \{1, \ldots, m\}$, $J \subset \{1, \ldots, n\}$, and $J + m \subset \{m+1, \ldots, m+n\}$. This is a bifunctor that is associative and exact in both factors. For notational purposes, we write S_I^n for the simple corresponding to the vertex x_I in Cube_n and P_I^n for the corresponding projective. An example:

$$\begin{tikzcd} 0 \arrow[r] & k \end{tikzcd} \boxtimes \begin{tikzcd} 0 \arrow[r] \arrow[d] & k \arrow[d] \\ 0 \arrow[r] & k \end{tikzcd} = \begin{tikzcd} & 0 \arrow[rr] \arrow[dd, dashed] & & k \arrow[dd] \\ 0 \arrow[ur] \arrow[rr] \arrow[dd] & & 0 \arrow[ur] \arrow[dd] & \\ & 0 \arrow[rr, dashed] & & k \\ 0 \arrow[ur, dashed] \arrow[rr] & & 0 \arrow[ur] & \end{tikzcd}$$

In general, a product of projectives is projective: $P_I^m \boxtimes P_J^n = P_{I\cup J+m}^{m+n}$. We will shift this product slightly by inserting a copy of the indecomposable projective $P_\emptyset^1$ of Cube_1:

$$\otimes : \mathrm{Cube}_m \times \mathrm{Cube}_n \to \mathrm{Cube}_{m+n+1} : (U_\bullet, V_\bullet) \mapsto U_\bullet \boxtimes P_\emptyset^1 \boxtimes V_\bullet.$$

Multiplying by $P_\emptyset^1$ has the effect of copying. This product is still associative and projectives again get mapped to projectives:

$$P_I^m \otimes P_J^n = P_{I\cup(J+(m+1))}^{m+n+1}$$

This gives a monoidal structure without unit on the category $\mathrm{Cube}_\bullet = \oplus_{n\geq 0}\mathrm{Cube}_n$ and also on the category $\mathrm{cube}_\bullet = \bigoplus_{n\geq 0}\mathrm{cube}_n$. These categories should be interpreted in the following way: the objects are formal direct sums $\bigoplus_i C_i$, with C_i in Cube_i, and the morphisms are given by

$$\mathrm{Hom}_{\mathrm{Cube}_\bullet}\left(\bigoplus_i C_i, \bigoplus_i D_i\right) = \prod_i \mathrm{Hom}_{\mathrm{Cube}_i}(C_i, D_i).$$

In the case of $\mathrm{cube}_\bullet$ we consider the full subcategory consisting of objects that have only a finite number of non-zero terms. This will be indicated by writing $\bigoplus'$. It seems rather artificial to have monoidal categories without unit. This will be fixed in the next section, by allowing negative values of n.

3.3 Cube$_\bullet$ and N-Koszulity

$N = 2$ case

The link between Cube and Koszul algebras originates in Backelin's theorem, and is quite general. We'll introduce the necessary lemma in the context of a general k-linear abelian category, where the notion of distributive collection of subobjects makes sense. The correct definitions and details of the categorical versions of these set-theoretic concepts can be found in Murfet [61].

Lemma 3.3.1 *A collection of subobjects $R_1, \ldots, R_n \subset C$ of an object in a k-linear abelian category $\mathcal{C}$ generates a distributive lattice if and only if there exists an exact functor $F : \mathrm{cube}_n \to \mathcal{C}$ such that $F(P_\emptyset^n) = C$ and $F(P_{\{i\}}^n \to P_\emptyset^n) = (R_i \hookrightarrow C)$.*

Proof: We will only consider the case where $\mathcal{C}$ is the category of vector spaces over a field. First of all, notice that F is unique because it is defined on all simple objects, by the existence of projective resolutions and right exactness. From the characterizations of projective covers for quivers, it is clear that the $P_{\{i\}}^n$ generate a distributive lattice in $P_\emptyset^n$. Since distributivity is preserved under exact functors, one direction follows immediately from the definition of F. To prove the other direction, we will use the third characterization of Proposition 3.1.5. Denote $[n] = \{1, \ldots, n\}$. First we'll construct an exact functor from cube_n to the category of complexes over $\mathcal{C}$. This sends an object $V_\bullet = (V_J)_{J\subset[n]}$ to the complex

$$C(V_\bullet)^0 \to C(V_\bullet)^1 \to \cdots \to C(V_\bullet)^n \to 0,$$

where

$$C(V_\bullet)^i = \bigoplus_J \bigoplus_{K\subset[n]} V_J \otimes (C/\sum_{j\in K} R_j),$$

and K has to satisfy $J \cup K = [n]$ and $|J \cap K| = i$. The differential is zero, unless $J \subset J'$, $K \subset K'$, and $|J' \backslash J| + |K' \backslash K| = 1$, in which case we have a map

$$d : V_J \otimes (C/\sum_{j \in K} R_j) \to V_{J'} \otimes (C/\sum_{j \in K'} R_j) : v \otimes c \mapsto (-1)^{|J| \cdot |K' \backslash K|} \phi_{J,J'}(v) \otimes \pi(c),$$

where π denotes the natural projection. For a simple object S_I, with $|I| = m$, these expressions become a great deal easier:

$$X/\sum_{j \in [n] \backslash I} R_j \to \oplus_{k \in I} X/\Big(\sum_{j \in [n] \backslash I} R_j + R_k\Big) \to \cdots \to X/\sum_{j \in [n]} R_j \to 0$$

Comparing this to Proposition 3.1.5 (3), we conclude that the complex associated to every simple in cube_n is acyclic in degree $i > 0$, and thus, the complex associated to any other object is too. This means that the functor

$$F : \mathrm{cube}_n \to \mathcal{C} : V_\bullet \mapsto H^0 C(V_\bullet)$$

is exact. It remains to be checked that $H^0 C(P^n_{\{i\}}) \cong R_i$, which is a straight forward verification. □

Notice that if $\mathcal{C}$ is a Grothendieck category, as is **Vect**, the category of all vector spaces over a field, then this functor extends to an exact functor from the category Cube_n, if we demand that it commutes with filtered colimits. Remember that in Grothendieck categories, filtered colimits are exact in the sense that for exact sequences

$$0 \to X_i \to Y_i \to Z_i \to 0,$$

for every i in a directed set I, the colimits (which exist by definition of a Grothendieck category) fit into the exact sequence

$$0 \to \mathrm{colim}\ X_i \to \mathrm{colim}\ Y_i \to \mathrm{colim}\ Z_i \to 0.$$

Thus the commutation requirement automatically defines F on infinite dimensional representations. With this lemma, one can deduce the following theorem.

Theorem 3.3.2 *The category of Koszul algebras is equivalent to the category of exact monoidal functors $F : (\mathrm{cube}_\bullet, \otimes) \to$ Vect, the category of finite dimensional vector spaces over k.*

Proof: From the product rule, it is immediately seen that $(S^0_\emptyset)^{\otimes n+1} = P^n_\emptyset$. Furthermore, there is an exact sequence

$$0 \to P^1_{\{1\}} \to P^1_\emptyset \to S^1_\emptyset \to 0,$$

so given an exact monoidal functor F, we can define

$$V := F(S^0_\emptyset),\ R := F(P^1_{\{1\}}) \subset F(P^1_\emptyset) \cong V \otimes V.$$

The algebra $TV/(R)$ is thus quadratic. It is also Koszul by Backelin's Theorem 3.1.6, since the images of the objects

$$(S^0_\emptyset)^{\otimes i-1} \otimes P^1_{\{1\}} \otimes (S^0_\emptyset)^{\otimes n-i} = P^n_{\{i\}} \subset P^n_\emptyset$$

are exactly the relation spaces and generate a distributive lattice due to the previous lemma. Given a Koszul algebra, we know from the lemma that there are exact functors

$$F_n : \mathrm{cube}_n \to \mathrm{Vect},\ P^n_\emptyset \mapsto V^{\otimes n+1},\ P^n_{\{i\}} \mapsto V^{\otimes i-1} \otimes R \otimes V^{\otimes n-i}$$

Remains to prove that the induced functor $F = \oplus_n F_n$ has a monoidal structure. To a $K_\bullet \in \mathrm{cube}_k$ and $L_\bullet \in \mathrm{cube}_l$, we associate the complexes $C(K_\bullet)$ and $C(L_\bullet)$ from the lemma above. Then there is a natural morphism of complexes

$$C(K_\bullet) \otimes C(L_\bullet) \to C(K_\bullet \otimes L_\bullet),$$

that is in fact a quasi-isomorphism. Since the F_n are exact, this induces an isomorphism

$$F_k(K_\bullet) \otimes F_l(L_\bullet) \to F_{k+l+1}(K_\bullet \otimes L_\bullet).$$

□

$N \geq 2$ case

To establish the connection with N-Koszul algebras, we modify the product $\otimes$ on $\mathrm{Cube}_\bullet$ again. For fixed N, the product is (notice we keep the notation $\otimes$):

$$\otimes : \mathrm{Cube}_m \times \mathrm{Cube}_n \to \mathrm{Cube}_{m+n+N-1} : (U_\bullet, V_\bullet) \mapsto U_\bullet \boxtimes P_\emptyset^{N-1} \boxtimes V_\bullet.$$

Associativity is not destroyed, and projectives still get sent to projectives:

$$P_I^m \otimes P_J^n = P_{I \cup (J+(m+N-1))}^{m+n+N-1}$$

In contrast to the previously defined product, this will be considered valid for all $m, n \geq -N+1$, providing us also with a unit object $P_\emptyset^{-N+1}$. Remember that for non-positive values of m, Q_m is a single vertex with no arrow, so there is only $P_\emptyset^m$ to consider. Written in full, the product for negative values is:

$$\begin{aligned}
\text{For } m \in \{-N+1, \ldots, 0\},\ n \geq 0: \quad P_\emptyset^m \otimes V_\bullet = \quad & P_\emptyset^{m+N-1} \boxtimes V_\bullet \\
\text{For } m \geq 0,\ n \in \{-N+1, \ldots, 0\}: \quad U_\bullet \otimes P_\emptyset^n = \quad & U_\bullet \boxtimes P_\emptyset^{n+N-1} \\
\text{For } m, n \in \{-N+1, \ldots, 0\}: \quad P_\emptyset^m \otimes P_\emptyset^n = \quad & P_\emptyset^{m+n+N-1}
\end{aligned}$$

In the same way as before, we denote $\mathrm{Cube}_\bullet = \bigoplus_{n \geq -N+1} \mathrm{Cube}_n$ and $\mathrm{cube}_\bullet = \bigoplus'_{n \geq -N+1} \mathrm{cube}_n$. To get more insight into the structure of these categories we give an explicit formula for the tensor product of simples in $\mathrm{Cube}_\bullet$. Let I and J be subsets of $[n]$ and $[m]$ respectively, and for non-positive m and n we set these to be $\emptyset$.

Proposition 3.3.3 *The Jordan-Hölder series for $S_I^m \otimes S_J^n$ is made up of $S_K^{m+n+N-1}$, for $K = I \cup J' \cup C$, where*

$$J' = J + m + N - 1,$$

$$C \subset \{\max(1, m+1), \ldots, \min(m+N-1, m+n+N-1)\}.$$

The ordering is according to the order of K ($S_{I \cup J'}^{m+n+N-1}$ on top), and all multiplicities are 1.

Proof: First consider $m, n \geq 1$. In this case $C \subset \{m+1, \ldots, m+N-1\}$. The effect of $S_I^m \boxtimes P_\emptyset^{N-1}$ is taking 2^{N-1} copies of S_I^m, labelling the vertices of each copy by succesive translations by m, and connecting corresponding vertices with proper orientation. This gives us a representation of a quiver having 2^{N-1} vertices of value k, and the rest 0. Taking the product of this with S_J^n gives us 2^{N-1} non-zero vertices. These occur exactly at the places that correspond to subsets $K \subset \{1, \ldots, m+n+N-1\}$ of the form

$$K = I \cup J' \cup C.$$

That they occur with multiplicity one is obvious from the above. For $m = -N+2, \ldots, 0, n \geq 1$, we have $I = \emptyset$, and by the definitions above, $S_\emptyset^m = P_\emptyset^m$, and thus

$$S_\emptyset^m \otimes S_J^n = P_\emptyset^{m+N-1} \boxtimes S_J^n.$$

The effect is taking 2^{m+N-1} copies of S_J^n just like above, with $m+N-1 \in \{1, \ldots, N-1\}$. This shows that we have to take $K = J' \cup C$, with $C \subset \{1, \ldots, m+N-1\}$. The other cases are completely similar, proving the proposition. $\square$

Having gained some insight into this new multiplication $\otimes$, dependent on N, we are now in a position to generalize the connection between Koszulity and Cube$_\bullet$. As usual, A denotes the algebra $TV/(R)$, where V is a finite dimensional k-vector space, and R is a subspace of $V^{\otimes N}$. Let us introduce some new notation:

$$R_i^n = V^{\otimes i-1} \otimes R \otimes V^{\otimes n-N-i+1} \subset V^{\otimes n}$$
$$R_I^n = \bigcap_{i \in I} R_i^n, \text{ for } I \subset \{1, \ldots, n-N+1\}$$
$$\mathcal{L}_n(A) = \text{the lattice generated by all } R_i^n, i \in \{1, \ldots, n-N+1\}.$$

Notice that with this notation, A is distributive iff each $\mathcal{L}_n(A)$ is a distributive lattice. Just like in the $N = 2$ case, we define a functor F_A by defining it on projectives, and demanding it to be right exact and commuting with filtered colimits

$$F_A : \text{Cube}_{n-N+1} \to \mathit{Vect} : P_I^{n-N+1} \mapsto R_I^n,$$

and the new product $\otimes$ we defined is such that this functor is monoidal. It is now time to digress a little on the extra technical condition we talked about in the intro to Section 3.1.

Definition 3.3.4 *The algebra A satisfies the extra condition if, for each n,*

$$R_i^n \cap R_j^n \subset R_{\{i+1,\ldots,j-1\}}^n,$$

where $i \in \{1, \ldots, n-2\}$, and $j \in \{i+2, \ldots, \min(i+N-1, n)\}$.

Notice that for $N = 2$, the condition is empty. To illustrate, in the case $N = 3$, this reduces to:

$$(V \otimes V \otimes R) \cap (R \otimes V \otimes V) \subset (V \otimes R \otimes V),$$

which is non-trivial, since it is not a distributivity relation. As we noticed before, Berger [12] proved that

$$\text{Distributivity} \ + \ \text{Extra condition} \ \Rightarrow \ N\text{-Koszul} \ \Rightarrow \ \text{Extra condition}.$$

The first implication will be proven later when we have fully developed the relation to $\mathrm{Cube}_\bullet$; the second is quite technical, and we refer to Berger's paper. What we would first like to show is that A is distributive if and only if the functor F_A is exact for all n. To do this, we try to find a complex in $\mathrm{Cube}_\bullet$ that gets mapped to the Koszul complex of A in Vect. This will allow for a study of A in the category $\mathrm{Cube}_\bullet$ which, as we have seen, is particularly well suited to homological calculations, being built out of representation categories of certain quivers with relations. For general N, the Koszul complex is defined by:

$$\cdots \longrightarrow A \otimes J_{2N} \xrightarrow{\delta^{N-1}} A \otimes J_{N+1} \xrightarrow{\delta} A \otimes R \xrightarrow{\delta^{N-1}} A \otimes V \xrightarrow{\delta} A \longrightarrow k,$$

where $J_n = R^n_{\{1,\ldots,n-N+1\}}$ if $n \geq N$, and $J_n = V^{\otimes n}$ if $n < N$. The exactness of this complex is equivalent to Definition 3.1.1, just like in the $N = 2$ case, and we refer to Berger [12] for the details. To establish the link with our monoidal category, we first look at the following objects in $\mathrm{Cube}_{m+n-N+1}$:

$$W_{m,n} = S^{m-N+1}_\emptyset \otimes S^{n-N+1}_{\{1,\ldots,n-N+1\}},$$

with the usual convention: if $n - N + 1 \leq 0$, then $\{1, \ldots, n - N + 1\} = \emptyset$.

Proposition 3.3.5 *For $m \neq 0$, there is a complex of the form*

$$\cdots \longrightarrow W_{m-2N,2N} \xrightarrow{\delta^{N-1}} W_{m-N-1,N+1} \xrightarrow{\delta} W_{m-N,N} \xrightarrow{\delta^{N-1}} W_{m-1,1} \xrightarrow{\delta} W_{m,0} \longrightarrow 0$$

that gets mapped to the degree m part of the Koszul complex. For $m = 0$, the same holds for the complex

$$\cdots \longrightarrow W_{-2N,2N} \xrightarrow{\delta^{N-1}} W_{-N-1,N+1} \xrightarrow{\delta} W_{-N,N} \xrightarrow{\delta^{N-1}} W_{-1,1} \xrightarrow{\delta} W_{0,0} \longrightarrow S^{-N+1}_\emptyset.$$

Proof: The object $W_{m,n}$ is the product of the 'extremal' simples. We have given a formula for the product of simples: in this case we get

$$(W_{m,n})_I = k \text{ iff } I = \{m+1, \ldots, m+n-N+1\} \cup C, \text{ and } C \subset \{\max(1, m-N+2), \ldots, m\}.$$

Since a presentation of $S^{m-N+1}_\emptyset$ is given by:

$$\bigoplus_{i \in \{1,\ldots,m-N+1\}} P^{m-N+1}_{\{i\}} \longrightarrow P^{m-N+1}_\emptyset \longrightarrow S^{m-N+1}_\emptyset \longrightarrow 0,$$

applying F_A yields

$$(R \otimes V^{\otimes m-N}) + (V \otimes R \otimes V^{\otimes m-N-1}) + \cdots + (V^{\otimes m-N} \otimes R) \longrightarrow V^{\otimes m} \longrightarrow F_A(S^{m-N+1}_\emptyset) \longrightarrow 0.$$

This allows us to conclude that $F_A(S^{m-N+1}_\emptyset) = A_m$. More easily, $F_A(S^{n-N+1}_{\{1,\ldots,n-N+1\}}) = F_A(P^{n-N+1}_{\{1,\ldots,n-N+1\}}) = J_n$, and so indeed

$$F_A(W_{m,n}) = A_m \otimes J_n.$$

Now we still have to find a nice map

$$\delta : W_{m,n} \to W_{m+1,n-1}.$$

The only natural way to do this is to take the identity if $(W_{m,n})_I = (W_{m+1,n-1})_I = k$ and zero otherwise. To show that this is a well defined map, it suffices to discard the following possibilities:

$$(W_{m,n})_I = (W_{m+1,n-1})_I = k, \text{ and}, (W_{m,n})_{I\cup\{l\}} = 0, \ (W_{m+1,n-1})_{I\cup\{l\}} = k,$$
$$(W_{m,n})_I = 0, \ (W_{m+1,n-1})_I = k, \text{ and}, (W_{m,n})_{I-\{l\}} = k, \ (W_{m+1,n-1})_{I-\{l\}} = k,$$

for then we would need that $k \xrightarrow{\mathrm{Id}} k \xrightarrow{\mathrm{Id}} k = k \xrightarrow{0} 0 \xrightarrow{0} k$, which isn't the case. This however, will not happen; let's consider the first case: if $(W_{m,n})_I = (W_{m+1,n-1})_I = k$, then by our formula

$$\{m+1, \ldots, m+n-N+1\} \subset I \subset \{m-N+3, \ldots, m+n-N+1\}.$$

It is now impossible to find an element l, such that $l \notin \{m-N+2, \ldots, m+n-N+1\}$, but $l \in \{m-N+3, \ldots, m+n-N+1\}$, which is what we would need for the bad cases to occur. The second case is completely similar, and so the map is well defined in $\mathrm{Cube}_\bullet$. It is also immediately clear that $F_A(\delta) = \delta_m$. Lastly, one sees that $\delta^N = 0$, for otherwise there would be an I such that

$$\{m+1, \ldots, m+n-N+1\} \subset I \subset \{m+2 = m+N-N+2, \ldots, m+n-N+1\},$$

which isn't possible. Notice that $\delta^{N-1} \neq 0$: take $I = \{m+1, \ldots, m+n-N+1\}$. All of this tells us that for $m \neq 0$ the complex

$$\cdots \longrightarrow W_{m-2N,2N} \xrightarrow{\delta^{N-1}} W_{m-N-1,N+1} \xrightarrow{\delta} W_{m-N,N} \xrightarrow{\delta^{N-1}} W_{m-1,1} \xrightarrow{\delta} W_{m,0} \longrightarrow 0$$

gets mapped to the degree m part of the Koszul complex, and if $m = 0$, the same holds for the complex

$$\cdots \longrightarrow W_{m-2N,2N} \xrightarrow{\delta^{N-1}} W_{m-N-1,N+1} \xrightarrow{\delta} W_{m-N,N} \xrightarrow{\delta^{N-1}} W_{m-1,1} \xrightarrow{\delta} W_{m,0} \longrightarrow S_\emptyset^{-N+1}.$$

□

We now calculate the homology of this complex in the category $\mathrm{Cube}_\bullet$.

Proposition 3.3.6 *The homology of the $(W_{m,n})$-complex is an extension of simples S_K^{m-N+1}, such that K contains 'holes' of size $\leq N-2$. A subset $I \subset \{1, \ldots, m\}$ is said to have a hole of size u, $u \geq 1$ if there exist $i < j \in I$ such that $j - i = u + 1$ and all l strictly between i and j do not belong to I.*

Proof: Consider the slightly more general problem of describing the homology of the complexes

$$W_{x,y} \xrightarrow{\delta^a} W_{x+a,y-a} \xrightarrow{\delta^{N-a}} W_{x+N,y-N},$$

for some x, y and $0 < a < N$. The case we are interested in requires $a = 1, N-1$. The homology is then $H = \mathrm{Ker}\ \delta^{N-a}/\mathrm{Im}\ \delta^a$, and by the computation of the $(W_{x,y})_I$ we did above, one can tell exactly when a H_I is non-zero. In full:

$$(W_{x,y})_I = k \text{ iff } \{x+1, \ldots, x+y-N+1\} \subset I \subset \{x-N+2, \ldots, x+y-N+1\},$$
$$(W_{x+a,y-a})_I = k \text{ iff } \{x+a+1, \ldots, x+y-N+1\} \subset I \subset \{x+a-N+2, \ldots, x+y-N+1\},$$
$$(W_{x+N,y-N})_I = k \text{ iff } \{x+N+1, \ldots, x+y-N+1\} \subset I \subset \{x+2, \ldots, x+y-N+1\}.$$

From this it follows that $H_I \neq 0$ if and only if

$$\{x+1, \ldots, x+y-N+1\} \not\subset I \not\subset \{x+2, \ldots, x+y-N+1\}, \text{ and}$$

I satisfies the third subset relation above. In that case, the situation $0 \to k \to 0$ arises. If I satisfies these subset relations, we see that there has to exist an $i \in I$, such that

$$x+a-N+2 \leq i \leq x+1.$$

If, moreover, we suppose that the inequality

$$N \leq y-a$$

holds, then none of the sets appearing in these relations are empty, and we conclude that there exists $j \notin I$ such that

$$x+1 \leq j \leq x+a.$$

This can be used to find holes in I. Look at the triple $\{i, j, x+a+1\}$. Then $i < j < x+a+1$, and we conclude

$$x+a-N+2 \leq i \Rightarrow -i \leq -x-a+N-2 \Rightarrow x+a+1-i \leq N-1.$$

Since $i, x+a+1 \in I$, I contains a hole of size $\leq N-2$. This is exactly what needed to be proved. Remains to consider the cases where $N > y-a$. Looking back at the $(W_{m,n})$-complex, this occurs only for $(x,y,a) = (m-N, N, N-1)$ and $(x,y) = (m-1, 1, 1)$.

- $(x,y,a) = (m-N, N, N-1)$: the conditions for non-zero homology H_I now become

  $$\{m-N+1, \ldots, m-N+1\} = \{m-N+1\} \not\subset I \not\subset \{m-N+2, \ldots, m-N+1\} = \emptyset,$$

  $$\{m, \ldots, m-N+1\} = \emptyset \subset I \subset \{m-N+1, \ldots, m-N+1\} = \{m-N+1\}.$$

  No I satisfies all three properties, so the homology is zero.

- $(x,y,a) = (m-1, 1, 1)$: plugging in our conditions for I, we see that $\emptyset \not\subset I \not\subset \emptyset$, making the homology zero again.

□

Having completely described the homology of the complex in $\mathrm{Cube}_\bullet$ that gets mapped to the Koszul complex, we can now describe the relevant properties of A in terms of F_A. Denote by LF_A the left derived functor of the right exact functor F_A. Remember that one computes this derived functor on an object by taking a projective resolution of the object, applying F_A to the resolution (giving a complex which is exact on the right because F_A is only right exact) and then taking homology. First we need a preparatory lemma

Lemma 3.3.7 *Given acyclic objects $M, N \in \mathrm{Cube}_\bullet$, that is L_iF_AM and L_iF_AN are zero for $i > 0$, then $M \otimes N$ is also acyclic.*

Proof: Since we are using quivers without loops or oriented cycles, we're in the setting of modules over finite-dimensional algebras, and we can take minimal projective resolutions $P_\bullet \to M \to 0$ and $Q_\bullet \to N \to 0$ of M and N respectively. In this case the tensor product bifunctor is exact, so $P_\bullet \otimes Q_\bullet$ is a minimal projective resolution of $M \otimes N$. Applying F_A to this resolution and using that it is monoidal we get a complex

$$F_A(P_\bullet) \otimes F_A(Q_\bullet) = F_A(P_\bullet \otimes Q_\bullet) \to F_A(M \otimes N) = F_A(M) \otimes F_A(N).$$

To compute $L_\bullet F_A(M \otimes N)$ we need the homology of this complex. Since both M and N are acyclic, we have resolutions

$$F_A(P_\bullet) \to F_A(M), \text{ and } F_A(Q_\bullet) \to F_A(N),$$

And so the $F_A(M) \otimes F_A(N)$ complex is also acyclic. □

Theorem 3.3.8 *For $A = TV/(R)$, and $R \subset V^{\otimes N}$, the following are equivalent:*

- *A is distributive,*
- *F_A is exact for all n*
- *All S_I^{n-N+1} are acyclic*

and concerning the extra condition, the following are also equivalent

- *A satisfies the extra condition*
- *For every n, and every I that contains a hole of size $\leq N-2$, $F_A(S_I^{n-N+1}) = 0$*

Proof: Looking back at Lemma 3.3.1 and the definition of F_A, we see that the equivalence of the first two statements is just a special case of that lemma (remember that we did not need the monoidal structure for this, so the value of N doesn't matter). The exactness of F_A obviously implies the acyclicity of the simples. Conversely, suppose all simples are acyclic, then the Jordan-Hölder theorem ensures the acyclicity of every object $M \in \mathrm{Cube}_\bullet$. This can be done by induction on the length of the composition series. For instance, suppose $0 \subset N \subset M$ is the composition series of M, then we have an exact sequence

$$0 \to N \to M \to M/N \to 0,$$

and N and M/N are acyclic. By a combination of the horseshoe lemma and the long exact homology sequence, we see that M is acyclic as well. Let us prove the second pair of equivalent conditions. Assume A satisfies the extra condition, and look at the presentation of S_I^{n-N+1} given by

$$\bigoplus_{l \notin I} P_{I \cup \{l\}}^{n-N+1} \to P_I^{n-N+1} \to S_I^{n-N+1} \to 0.$$

The right exactness of F_A tells us that

$$F_A(S_I^{n-N+1}) = \mathrm{coker}(\sum_{l \notin I} R_{I \cup \{l\}}^n \to R_I^n) = R_I^n / \sum_{l \notin I} R_{I \cup \{l\}}^n$$

If I is as in the conditions, and i and j denote the boundaries of the hole of I, there exists $i < l < j$, and the extra condition says

$$R_i^n \cap R_j^n \subset R_l^n,$$

concluding that $R_I^n \subset R_{I\cup\{l\}}^n$ and thus $R_I^n = R_{I\cup\{l\}}^n$. Along with the computation above, this implies $F_A(S_I^{n-N+1}) = 0$. Conversely, by the same reasoning as before, we have that for an I as in the statement

$$R_I^n = \sum_{l\notin I} R_{I\cup\{l\}}^n.$$

Continuing this process for every l such that $I \cup \{l\}$ has a hole of size $\leq N-2$, we eventually get

$$R_I^n = \sum_K R_K^n,$$

where $I \subset K \subset \{1,\ldots,n\}$ and K has no holes of size $\leq N-2$. Given $1 \leq i < j \leq i+N-1$, take an l between i and j. Taking $I = \{i,j\}$, every K that arises from the above process will contain l, and so $R_K^n \subset R_l^n$, and also $R_i^n \cap R_j^n = R_I^n \subset R_l^n$, and the extra condition holds. □

If A satisfies the extra condition, even more can be said about the simples. To do this, we need a general property of finitely filtered complexes in abelian categories.

Definition 3.3.9 *A complex $C_\bullet$ in an abelian category $\mathcal{A}$ is finitely filtered if each object C_n in the complex is filtered (i.e. has a filtration by subobjects), and the differential respects this filtration, giving us a sequence*

$$0 = F_0C_\bullet \subset F_1C_\bullet \subset \cdots \subset F_nC_\bullet = C_\bullet.$$

Note that each of the $F_kC_\bullet$ is a complex in $\mathcal{A}$.

Lemma 3.3.10 *Suppose $C_\bullet$ is a finitely filtered complex in an abelian category $\mathcal{A}$. If the associated graded complex $\mathrm{gr}_F C_\bullet = \oplus_i F_{i+1}C_\bullet/F_iC_\bullet$ is acyclic, then $C_\bullet$ is acyclic too.*

Proof: The short exact sequences of complexes

$$0 \to F_iC_\bullet \to F_{i+1}C_\bullet \to F_{i+1}C_\bullet/F_iC_\bullet \to 0$$

induce long exact sequences on homology

$$\cdots \to H_{n+1}(F_{i+1}C_\bullet/F_iC_\bullet) \to H_n(F_iC_\bullet) \to H_n(F_{i+1}C_\bullet) \to H_n(F_{i+1}C_\bullet/F_iC_\bullet) \to \cdots,$$

which by assumption, becomes

$$\cdots 0 \to H_n(F_iC_\bullet) \to H_n(F_{i+1}C_\bullet) \to 0 \to \cdots.$$

By induction on i, this now proves the claim. □

These kind of arguments, i.e. using the homology of the associated graded to study the homology of the original complex, give rise to the theory of spectral sequences, which we prefer to circumvent here, see Weibel [80], Chapter 5.

Theorem 3.3.11 *Assume A satisfies the extra condition, then*

- *For every n and every I that contains a hole of size $\leq N-2$, the simple S_I^{n-N+1} is acyclic.*

In that case the following are equivalent conditions

- *A is N-Koszul*
- *For every m, the simple $S_\emptyset^{m-N+1}$ is acyclic*

Proof: Given an I as in the conditions, take some corresponding i, l, j. As we have seen, the terms in the minimal projective resolution of S_I^{n-N+1} are all of the form $\bigoplus_{K\supset I} P_K^{n-N+1}$. Among those sets K, there will be those not containing l and those containing l. Given a set of the first kind K_1, obviously $K_1 \cup \{l\}$ is a set of the second kind, and vice versa, given a K_2 of the second kind, $K_2 \backslash \{l\}$ is of the first kind, so the K's come in pairs $K_1, K_2 = K_1 \cup \{l\}$. Since the K_1's do not contain l, they still contain a hole of size $\leq N-2$, and the same reasoning as in the above proof shows that

$$F_A(P_{K_1}^{n-N+1}) = R_{K_1}^n = R_{K_2}^n = F_A(P_{K_2}^{n-N+1}).$$

If we denote by $K_\bullet$ this minimal projective resolution of S_I^{n-N+1}, then we need to show that the complex $C_\bullet = F_A(K_\bullet)$ is acyclic. To use the lemma, we first put a filtration on $C_\bullet$ by defining

$$F_i C^\bullet = \bigoplus_K F_A(P_K^{n-N+1}),$$

where K is taken as before, and such that $|K\backslash\{l\}| \leq i$. Given a non-zero map $P_K^{n-N+1} \to P_L^{n-N+1}$, we know that $L \subset K$, and $F_i C_\bullet$ is thus closed under the differential. This provides us with a finite filtration. Remains to check that $\mathrm{gr}_F C_\bullet$ is acyclic. Notice that because $F_i C_\bullet$ is not just a subset of $F_{i+1} C_\bullet$, but even a direct summand, there is an isomorphism of graded vector spaces $\mathrm{gr}_F C_\bullet \cong C_\bullet$. The only non-zero maps in $\mathrm{gr}_F C_\bullet$ are of the form $F_A(P_{K_2}^{n-N+1}) \to F_A(P_{K_1}^{n-N+1})$, for K_1 and K_2 as before. These maps are just the identity, possibly modulo a sign, and so $\mathrm{gr}_F C_\bullet$ is acyclic.

To prove the second part of the theorem, first assume that A is N-Koszul. We proceed by induction on m. The $m = 0$ case is trivial, so take $m > 0$. Remember that the complex

$$\cdots \longrightarrow W_{m-2N,2N} \xrightarrow{\delta^{N-1}} W_{m-N-1,N+1} \xrightarrow{\delta} W_{m-N,N} \xrightarrow{\delta^{N-1}} W_{m-1,1} \xrightarrow{\delta} S_\emptyset^{m-N+1} \longrightarrow 0$$

gets mapped to the degree m part of the Koszul complex. For $a > 0$, we prove that $W_{m-a,a}$ is acyclic. In Proposition 3.3.6 we proved that $W_{m-1,1}$ is acyclic. Assume $a > 1$, then

$$W_{m-a,a} = S_\emptyset^{m-a+2-N} \otimes P_{\{1,\ldots,a-N\}}^{a-N}.$$

The first factor is acyclic because $m - a < m$, and we can use the induction hypothesis on m, and the second factor is projective, thus also acyclic. By Lemma 3.3.7, $W_{m-a,a}$ is also acyclic. If the above complex were to have no homology, and remembering that left derived functors can be computed from acyclic resolutions, we could use this complex to calculate $L_i F_A S_\emptyset^{m-N+1}$. The complex does have homology, but the same conclusion can be drawn. By Proposition 3.3.6 we know that the homology of the complex is an extension of simples having holes of size $\leq N-2$, and by the first part of the theorem we're proving, these simples are acyclic. We can use the same Jordan-Hölder argument as in Theorem 3.3.8 to see that the homology of the complex is acyclic. Moreover, by the second part of the same theorem, this homology gets killed by F_A. It is standard that left derived functors can be computed from acyclic resolutions; in this case however the complex is not a resolution. Given a complex $A_\bullet \to M$, made up out of acyclic objects having acyclic

homology that gets mapped to zero by a right exact functor F, the left derived functors L_iFM can be computed starting from this complex, without it having to be a resolution. This is proven by a standard diagram chase, starting from a projective resolution $P_\bullet$ of M, giving a map of complexes $F(P_\bullet) \to F(A_\bullet)$, then beginning the chase. This is exactly what we need to compute the derived functors from

$$\cdots \to F_A W_{m-N,N} \to F_A W_{m-1,1} \to S_\emptyset^{m-N+1} \to 0.$$

The reason we introduced this complex a while ago was exactly because it got mapped to the degree m part of the Koszul complex, and because we start with an N-Koszul algebra, we know this is exact, and $S_\emptyset^{m-N+1}$ is acyclic.

Conversely, if all $S_\emptyset^{m-N+1}$ are acyclic, then the same reasoning as before tells us that the terms of the sequence

$$\cdots \longrightarrow W_{m-2N,2N} \xrightarrow{\delta^{N-1}} W_{m-N-1,N+1} \xrightarrow{\delta} W_{m-N,N} \xrightarrow{\delta^{N-1}} W_{m-1,1} \xrightarrow{\delta} S_\emptyset^{m-N+1} \longrightarrow 0$$

are acyclic, the homology of the sequence is acyclic, and F_A kills the homology. Thus the left derived functors can again be computed from this sequence. Since $S_\emptyset^{m-N+1}$ is acyclic, the left-derived functors are zero, and the complex becomes exact after applying F_A. Again, this is exactly the degree m part of the Koszul complex, and gathering all m, this tells us that A is N-Koszul. □

Having obtained an abstract theoretical framework for studying N-Koszulity, we can now give an easy proof of Backelin's theorem.

Theorem 3.3.12 (Backelin's theorem, alternative proof) *If $N = 2$, then A is Koszul if and only if A is distributive.*

Proof: Assuming A is Koszul, Theorem 3.3.8 tells us it is sufficient to prove that all S_I^{n-1} are acyclic. Since the extra condition is vacuous in the $N = 2$ case, we already know from Theorem 3.3.11 that the $S_\emptyset^{n-1}$ are acyclic, and we can use induction on the sum $n + |I|$. Given an I of order $|I| \geq 1$, and an element $l \in I$, set

$$I_1 = I \cap \{1, \ldots, l-1\}$$

$$I_2 = (I \cap \{l+1, \ldots, n-1\}) - l$$

The orders of these subsets satisfy $|I_1| + |I_2| = |I| - 1$, and they allow for an exact sequence

$$0 \to S_I^{n-1} \to S_{I_1}^{l-1} \otimes S_{I_2}^{n-l-1} \to S_{I\setminus\{l\}}^{n-1} \to 0,$$

because by Proposition 3.3.3, there are only two simples occurring in the Jordan-Hölder series of $S_{I_1}^{l-1} \otimes S_{I_2}^{n-l-1}$. Since

$$|I_1| + l + |I_2| + (n-l) = |I| + (n-1),$$

we can use induction on the second term, and obviously also the third, so they are acyclic (we also used Lemma 3.3.7). By applying F_A, we get an exact sequence

$$\cdots \to 0 \to L_1 F_A S_I^{n-1} \to 0 \to 0 \to F_A S_I^{n-1} \to F_A(S_{I_1}^{l-1} \otimes S_{I_2}^{n-l-1}) \to F_A S_{I\setminus\{l\}}^{n-1} \to 0,$$

proving that S_I^{n-1} is acyclic as well.

Conversely, if A is distributive, Theorem 3.3.8 tells us that all S_I^{n-1} are acyclic, and by Theorem 3.3.11 and the fact that the extra condition is vacuous in the $N = 2$ case, this is sufficient to prove Koszulity. $\square$

Notice that the easy second implication of this proof works for arbitrary N if we also assume the extra condition, thus proving Berger's [12] implication

$$\text{Distributivity} \ + \ \text{Extra condition} \ \Rightarrow \ N\text{-Koszul.}$$

3.4 Worked example: Yang-Mills algebras

One of our motivations for introducing distributive algebras has been to study generalizations of Koszul algebras in the cases where the relations are cubic, quartic, or higher in degree. In this section, we will present a worked out example of an algebra that is 3-Koszul and distributive. In the process, we will explain the only method known at present to determine distributivity of a given algebra. The algebras we will consider are deformed Yang-Mills algebras.

Definition 3.4.1 *The deformed Yang-Mills algebra $A_\lambda, \lambda \in k$, where* $\mathrm{char}(k) = 0$ *is defined as*

$$A_\lambda = k\langle x_1, \ldots, x_n\rangle / \big(\sum_{j,p} g^{jp}([x_j, [x_i, x_p]] + \lambda\{x_i, x_j x_p\}) \mid i \in \{1, \ldots, n\}\big),$$

where the $g^{ij}, i, j \in \{1, \ldots, n\}$ denote the components of the inverse matrix of a real, symmetric (and of course invertible) matrix $(g_{ij})_{i,j}$.

Some background: Yang-Mills theory is a non-abelian gauge theory based on the Lie group $SU(n)$, and is used by physicists in the description of the standard model. Classic references are Rubakov [69] and Frampton [31]. A lot of solutions to the equations governing this theory are characterized by topological quantities, so called instantons, which is why the theory of fibre bundles was brought in. In this language, gauge potentials become coordinates of a connection form in a principal fibre bundle, and the Yang-Mills equations then become equations for these connections. Taking as base space a pseudo-Euclidean space $(\mathbb{R}^{s+1}, (g_{ij})_{i,j})$, where the matrix is invertible, real and symmetric (not necessarily positive definite), with coordinates y^i, any complex vector bundle of rank p is isomorphic to a trivial bundle $\mathbb{C}^p \otimes \mathbb{R}^{s+1} \to \mathbb{R}^{s+1}$. A connection on such a bundle is then just a matrix valued 1-form $A_i dy^i$, $(A_{ij})_{ij} \in M_p(\mathbb{C})$. The covariant derivative corresponding to this matrix is $\nabla_i = \frac{\partial}{\partial y^i} + A_i$, and the Yang-Mills equations are then

$$g^{jp}[\nabla_j, [\nabla_i, \nabla_p]] = 0, \ i \in \{1, \ldots, n\},$$

where $(g^{ij})_{ij}$ is inverse to $(g_{ij})_{ij}$. It is obvious that any solution to these equations carries a representation of the algebra A_0 defined above. From a physical standpoint, the impetus to study deformations of this algebra comes from string theory, see [59, 62]. To us, it is obvious that A_0 plays a fundamental role, and one could hope that studying this algebra abstractly could lead to interesting mathematics in the same way that the Heisenberg group does, see for example Howe [45]. The deformed versions are then just considered as extras, that can be taken into account using the same formalism.

The techniques we will use to study these algebras originate in string theory, keywords being mirror symmetry, quiver gauge theories and super potentials. For an introduction to super potentials in mathematics,

and the connection between algebras derived from such potentials and Calabi-Yau algebras, see Van den Bergh [77, 78] and Ginzburg [38]. For a vector space V of dimension $n \geq 2$, consider $TV = k\langle x_1, \ldots, x_n\rangle$, and a monomial $m \in TV$. The circular or cyclic derivative of m with respect to x_i is defined to be

$$\frac{{}^\circ\partial m}{\partial x_i} = \sum_{m = ux_iv} vu,$$

and is extended to a linear map

$$\frac{{}^\circ\partial}{\partial x_i} : TV/[TV, TV] \to TV,$$

where $[TV, TV]$ is the k-subspace of TV spanned by commutators. It is obvious that this notion is invariant under cyclic permutations of the variables, hence the name. A super potential is then a homogeneous element $w \in TV/[TV, TV]$ of degree ≥ 3.

Definition 3.4.2 *Given a super potential w, the Jacobian algebra associated to w is the algebra*

$$A = k\langle x_1, \ldots, x_n\rangle / \big(\frac{{}^\circ\partial w}{\partial x_i} \mid i \in \{1, \ldots, n\}\big)$$

If V is equipped with an actual Euclidean structure $g_{ij} = (x_i, x_j)$, (like above, though now we assume positive definiteness!) and $O(V)$ is the corresponding orthogonal group, then the motivation for choosing the specific deformations A_λ of A_0 is given by

Lemma 3.4.3 *The space of $O(V)$-invariants $(TV/[TV, TV])_4^{O(V)}$ is spanned by the elements*

$$w_1 = \sum_{i,j,p,q} g^{ip} g^{jq} [x_i, x_j][x_p, x_q], \text{ and } w_2 = \Big(\sum_{i,j} g^{ij} x_i x_j\Big)^2$$

Proof: Assume k is algebraically closed, and a basis is chosen such that $(g_{ij})_{i,j} = (\delta_{ij})_{i,j}$. The first fundamental theorem of classical invariant theory for the orthogonal group, see Procesi's excellent textbook [68], says that $(V^{\otimes 4})^{O(V)}$ is spanned by the elements

$$s_1 = \sum_{i,j} x_i x_i x_j x_j$$
$$s_2 = \sum_{i,j} x_i x_j x_i x_j$$
$$s_3 = \sum_{i,j} x_i x_j x_j x_i$$

which are obviously linearly independent. Now, as an algebraic group, $O(V)$ is reductive (see Chapter 5 or Humphreys [46]). This implies that the category of finite dimensional representations is semisimple, thus every $O(V)$-invariant surjective map splits, making the quotient map

$$(V^{\otimes 4})^{O(V)} \to (TV/[TV, TV])_4^{O(V)}$$

surjective. Since we're now working modulo $[TV, TV]$,

$$s_1 - s_3 = \sum_{i,j} [x_i, x_i x_j x_j] = 0,$$

while s_1 and s_2 are still independent. The result now follows since $w_1 = 2(s_2 - s_3)$ and $w_2 = s_1$. $\square$

The lemma tells us that the $O(V)$-invariant super potentials are exactly given by the $w_1 + \lambda w_2, \lambda \in k$. Then the Jacobian algebra associated to $w_1 + \lambda w_2$ is exactly the Yang-Mills algebras A_λ we defined before. This is an annoying computation, the main ingredient being

$$\frac{{}^\circ\partial}{\partial x_i}([x_0, x_1][x_2, x_3]) = \frac{{}^\circ\partial x_0}{\partial x_i}[x_1, [x_2, x_3]] + \frac{{}^\circ\partial x_1}{x_i}[[x_2, x_3], x_0] + \frac{{}^\circ\partial x_2}{\partial x_i}[x_3, [x_0, x_1]] + \frac{{}^\circ\partial x_3}{x_i}[[x_0, x_1], x_2]$$

Confluence

Before proving that the A_λ are distributive, we will discuss 'confluence'. This is a concept introduced by Berger [12] to establish distributivity of an algebra if $N > 2$. Suppose W is a finite dimensional vector space of dimension n, with totally ordered basis $(e_i)_i$.

Definition 3.4.4 *Fix a subspace $U \subset W$. A basis element e_j of W is non-reduced with respect to U if there exist coefficients c_i such that*

$$e_j - \sum_{i<j} c_i e_i \in U.$$

The set of non-reduced basis elements is denoted $\mathrm{NRed}(U)$.

The number of elements in $\mathrm{NRed}(U)$ is exactly the dimension of U: if this wouldn't be the case, suppose $|\mathrm{NRed}(U)| = p$, and U contains elements

$$e_{j_1} - \sum_{i<j_1} c_{1i} e_i,\ e_{j_2} - \sum_{i<j_2} c_{2i} e_i, \ldots, e_{j_p} - \sum_{i<j_p} c_{pi} e_i.$$

We can further assume that $e_{j_1} < e_{j_2} < \cdots < e_{j_p}$. If $\dim(U) < p$, then one of these vectors has to be a linear combination of previous ones, which is obviously impossible, so $\dim(U) \geq p$, which is impossible, since by rescaling, every basis vector of U can be turned into an element of $\mathrm{NRed}(U)$. It is also obvious that $\mathrm{NRed}(U') \subset \mathrm{NRed}(U)$ if $U' \subset U$, from which it follows that

$$\mathrm{NRed}(U \cap U') \subset \mathrm{NRed}(U) \cap \mathrm{NRed}(U')$$
$$\mathrm{NRed}(U + U') \supset \mathrm{NRed}(U) \cup \mathrm{NRed}(U')$$

Berger proved that

$$\mathrm{NRed}(U \cap U') = \mathrm{NRed}(U) \cap \mathrm{NRed}(U') \Leftrightarrow \mathrm{NRed}(U + U') = \mathrm{NRed}(U) \cup \mathrm{NRed}(U'),$$

which is precisely the content of

Definition 3.4.5 *The couple $U, U' \subset W$ is confluent if one, and thus both, of these inclusions are an equality.*

Berger then goes on to show the theorem

Theorem 3.4.6 *For a collection $(R_k)_k$ of pairwise confluent subspaces in W, $\mathrm{NRed}(-)$ defines an isomorphism between the lattice of subspaces $\mathcal{L}$ generated by the R_i and the lattice of subsets of $\{e_i \mid i = 1, \ldots, n\}$, generated by the images $\mathrm{NRed}(R_i)$. Thus $\mathcal{L}$ is distributive.*

Definition 3.4.7 *For $A = TV/(R)$ an N-homogeneous algebra, and $(e_i)_i$ a fixed totally ordered basis of V, A is said to be confluent if for all k the subspaces*

$$(R \otimes V^{k-N}),\ (V \otimes R \otimes V^{\otimes k-N-1}), \ldots, (V^{\otimes k-N} \otimes R) \subset V^{\otimes k}$$

are pairwise confluent.

Using the theorem, it is obvious that confluence of A implies that A is a distributive algebra. To make it a little easier, Berger also proves that to establish confluency of A it is necessary and sufficient to prove that $R \otimes V^{\otimes i}$ and $V^{\otimes i} \otimes R$ are confluent inside $V^{\otimes i+N}$ for $i \in \{1, \ldots, N-1\}$. Now we can prove the theorem.

Theorem 3.4.8 *The deformed Yang-Mills algebras A_λ are distributive and satisfy the extra condition.*

Proof: We can assume k is algebraically closed and by the Lefschetz principle, set $k = \mathbb{C}$. Suppose $\{e_1, \ldots, e_n\}$ is an orthonormal basis of V. Depending on the parity of n, we can base change to

$$\left\{ \begin{array}{lcl} e'_k & = & \frac{1}{2}(e_{2k-1} + ie_{2k}) \\ e'_{m+k} & = & \frac{1}{2}(e_{2(m-k)+1} - ie_{2(m-k)+2}) \end{array} \right. \quad \left\{ \begin{array}{lcl} e'_k & = & \frac{1}{2}(e_{2k-1} + ie_{2k}) \\ e'_{m+1} & = & e_{2m+1} \\ e'_{m+k+1} & = & \frac{1}{2}(e_{2(m-k)+1} - ie_{2(m-k)+2}) \end{array} \right.$$

for $k \in \{1, \ldots, m\}$, and $n = 2m$, respectively $n = 2m+1$, so we can assume $(g^{ij})_{i,j}$ has the form

$$\begin{pmatrix} 0 & 0 & \cdots & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 \\ 1 & \cdots & 0 & 0 \end{pmatrix}$$

From now on we assume $(x_1, \ldots, x_n)$ is such an (ordered) basis. Putting $\bar{i} = n + 1 - i$, the equations of A_λ are

$$\sum_j [x_j, [x_i, x_{\bar{j}}]] + \lambda\{x_i, x_j x_{\bar{j}}\},$$

and by expanding, we see that

$$\{x_n x_i x_1 \mid i = 1, \ldots, n\} \subset \mathrm{NRed}(R).$$

Since $\dim(R) = n$, we have equality. To check confluency, we only need to prove the equalities

$$\mathrm{NRed}(R_1^4 \cap R_2^4) = \mathrm{NRed}(R_1^4) \cap \mathrm{NRed}(R_2^4)$$
$$\mathrm{NRed}(R_1^5 \cap R_3^5) = \mathrm{NRed}(R_1^5) \cap \mathrm{NRed}(R_3^5)$$

Since the inclusion $\subset$ comes for free, it suffices to prove that

$$\dim(R_1^n \cap R_{n-2}^n) \geq |\mathrm{NRed}(R_1^n) \cap \mathrm{NRed}(R_{n-2}^n)|,$$

for both cases $n = 4, 5$. If $n = 4$, then from $\mathrm{NRed}(R)$ we deduce that

$$\mathrm{NRed}(R_1^n) \cap \mathrm{NRed}(R_{n-2}^n) = \{x_n x_n x_1 x_1\}.$$

Thus it suffices to find a non-zero element in $R_1^4 \cap R_2^4$: for this one can take

$$\bar{w}_\lambda = \sum_i x_i \frac{{}^\circ\partial w}{\partial x_i} = \sum_i \frac{{}^\circ\partial w}{\partial x_i} x_i,$$

where $w = w_1 + \lambda w_2$ is the deformed super potential. In the case $n = 5$, the RHS is empty so there is nothing to prove. For the extra condition, it suffices by Theorem 3.4.6 to check the inclusion

$$\mathrm{NRed}(R_1^5) \cap \mathrm{NRed}(R_3^5) \subset \mathrm{NRed}(R_2^5),$$

which is obvious since the LHS is empty. □

Corollary 3.4.9 *The deformed Yang-Mills algebras are* 3*-Koszul.*

In general, for any reductive group G one can look for algebras derived from G-invariant super potentials and get an algebra with a nice symmetry group. An example for algebraic groups of type G_2 can be found in Smith [71], where the super potential is derived from a generic 7-dimensional form of degree 3 that is stabilized by G_2. Another worked example of a 3-Koszul and distributive algebra can be found in Van den Bergh and Kriegk [51], in the setting of AS-regular algebras.

4

Endomorphism quantum groups

In this chapter we will construct a non-commutative version of the coordinate ring of endomorphisms acting on a vector space, and use the theory built up in the previous chapter to study its representation theory. In '86, Drinfeld [23] famously introduced quantum groups after extensive work by a group of Russian mathematical physicists, focussing on the quantum inverse scattering method. Though initially motivated by physics, gradually applications were found in diverse areas of mathematics, ranging from knot theory to differential equations to 3-manifold invariants, see Kassel [49]. The methods developed were rather ad hoc, and it was Manin who, in his '88 paper [55], proposed to study the newly developed quantum groups as symmetry 'groups' of non-commutative spaces, reflecting the commutative theory. Manin developed the theory for quadratic algebras, and was able to obtain the familiar GL_q and SL_q Hopf algebras, by solving a universality question, and 'breaking the symmetry' by dividing out extra relations. Our interest lies primarily in the universal object, not in the connection with the known quantum groups. It is our hope that these objects can be used to study the underlying quadratic algebra, since their universal property ensures functoriality, which could provide a new way of studying invariants associated to the algebras under consideration.

4.1 The universal coacting bialgebra

Though we are eventually concerned with distributive algebras, for the first part we will consider A of the form $TV/(R)$, for $R \subset V^{\otimes N}$. Notice this case is more general than Manin's. The universal coacting bialgebra is an algebra B equipped with an algebra morphism

$$\delta : A \to B \otimes A,$$

that respects the grading $\delta(A_n) \subset B \otimes A_n$, such that for any algebra C, and any algebra morphism

$$\partial : A \to C \otimes A,$$

that preserves the grading, there exists a unique algebra morphism $\gamma : B \to C$ that makes the following diagram commute

$$\begin{tikzcd} A \arrow[r, "\partial"] \arrow[dr, "\delta"'] & C \otimes A \\ & B \otimes A \arrow[u, dashed, "\gamma \otimes A"'] \end{tikzcd}$$

The universality even makes B into a bialgebra, and A into a B-comodule algebra.

Proposition 4.1.1 *Given A as described, and suppose there exists an algebra B satisfying the above universal property, then B is a bialgebra, and A is a B-comodule algebra. Any other coaction of a bialgebra on A as an algebra is the push-out of this one.*

Proof: The comultiplication Δ is defined by considering the following commutative diagram:

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[d, "\delta"] & B \otimes A \arrow[d, dashed, "\Delta \otimes A"] \\ B \otimes A \arrow[r, "B \otimes \delta"] & B \otimes B \otimes A \end{tikzcd}$$

where $\Delta : B \to B \otimes B$ is the algebra morphism coming from the universality property with respect to the algebra morphism

$$(1_B \otimes \delta) \circ \delta : A \to B \otimes B \otimes A.$$

The counit ϵ comes from considering:

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[dr, "A"] & B \otimes A \arrow[d, dashed, "\epsilon \otimes A"] \\ & A \cong k \otimes A \end{tikzcd}$$

where ϵ is created by universality with respect to the algebra morphism

$$1_A : A \to k \otimes A \cong A$$

That these operations define a bialgebra structure on B has to be checked; this is not hard but requires rather big commutative diagrams that we feel do not benefit the general understanding. The interested reader can consult Pareigis' notes [63]. Finally, for any other coaction ∂ of a bialgebra C on A as an algebra (A is a C-comodule algebra), by universality there is an induced algebra map $\pi : B \to C$, and the coaction of C has the form $(\pi \otimes A) \circ \delta$. Using the universal property on the pair $(C \otimes C, (\Delta \otimes A) \circ \partial) = (C \otimes C, (A \otimes \partial) \circ \partial)$ and on $(k, (\epsilon \otimes A) \circ \delta)$ this is seen to be a bialgebra morphism. $\square$

Notice that we could have done this proof for A any k-algebra. The existence of the bialgebra B is slightly trickier in general. If such a B exists, we will denote it $\underline{\mathrm{end}}(A)$, to make the analogy with the commutative setting. We briefly outline a possible construction for a connected, locally finite dimensional $\mathbb{Z}$-graded algebra, based on a private communication with Van den Bergh. Pick a homogeneous basis $(x_i)_i$ for A

(possibly infinite) and assume for simplicity that $x_0 = 1$. Given any unital algebra morphism preserving the grading $\delta : A \to C \otimes A$, then for i, j such that $\deg(x_i) = \deg(x_j)$, one has

$$\delta(x_i) = \sum_{j \in J} z_i^j \otimes x_j, \ z_i^j \in C, \ |J| < \infty.$$

Since the $(x_i)_i$ form a basis, we also have structure constants

$$x_k x_l = \sum_{m \in M} c_{k,l}^m x_m, \ c_{k,l}^m \in k, \ |M| < \infty.$$

If we apply δ to this last equality, relations between the different z_i^j will appear. More precisely, for appropriate value of k, l and p:

$$\sum_{u,v} c_{u,v}^p z_k^u z_l^v = \sum_h c_{k,l}^h z_h^p.$$

Since δ is unital, this gives another relation on the z_i^j:

$$z_0^j = 1_C \delta_{0,j},$$

where we used the Kronecker delta. Motivated by this, take the algebra generated by the formal variables $(z_i^j)_{i,j}$, subject to these relations. The coaction and bialgebra structure are given by the formulas:

$$\begin{aligned} \delta(x_i) &= \sum_k z_i^k \otimes x_k \\ \Delta(z_i^j) &= \sum_k z_i^k \otimes z_k^j \\ \epsilon(z_i^j) &= \delta_{i,j} \end{aligned}$$

where we used the Kronecker delta in the last formula again, and all formulas respect the grading. This clearly gives the right universal property. Though the construction depends on the basis $(x_i)_i$, universality will guarantee it is well-defined.

A more categorical viewpoint is presented in Pareigis' (other) notes [64], where one can also find Tambara's proof for finite-dimensional A. In the N-homogeneous setting, we can take $z_i^j = x_j^* \otimes x_i$, where the $(x_i)_i$ form a basis for the finite dimensional vector space V, and the description of $\underline{\mathrm{end}}(A)$ is reduced to

$$\underline{\mathrm{end}}(A) = T(V^* \otimes V)/(\pi_N(R^\perp \otimes R)),$$

where π_N denotes the appropriate shuffle map, and $R^\perp$ is defined as in Chapter 1 by

$$R^\perp = \{r \in V^* \otimes V^* \mid r(R) = 0\}.$$

For a more detailed analysis of this situation, based on the internal cohomomorphism concept, see Polishchuk and Positselkski [66], Chapter 3, §4. By setting $\deg(z_i^j) = 1$, $\underline{\mathrm{end}}(A)$ becomes a graded algebra. The n-th graded piece, which we denote by $\underline{\mathrm{end}}(A)_n$ is a (finite dimensional) coalgebra itself by the restrictions of Δ and ϵ.

4.2 Representation theory of universal bialgebras

In the rest of this thesis, we will be interested in the representation theory of $\underline{\mathrm{end}}(A)$, as a first step towards understanding its role in the study of A. Since we are in the algebraic setting, we will be interested in comodules of $\underline{\mathrm{end}}(A)$. That these are more fundamental than the modules can also be motivated from the theory of algebraic groups, where a rational module for such a group is essentially a comodule for the coordinate algebra. To initiate this study, we review some basic facts about coalgebras. The dual of a finite dimensional algebra A is a coalgebra, and the dual of a finite dimensional coalgebra C is an algebra. The basic fact underlying this duality is that for a finite dimensional vector space V:

$$(V \otimes V)^* \cong V^* \otimes V^*,$$

in a natural way. The analogy goes further, since

Theorem 4.2.1 *For a finite dimensional coalgebra C, there is a functor (switching left and right structures!)*

$$\mathrm{CoMod}(C) \to \mathrm{Mod}(C^*)$$

providing an equivalence of categories.

shows that coalgebras only provide for a restatement of the theory of finite dimensional algebras. In the infinite dimensional setting however, the dual of an algebra is not necessarily a coalgebra, and one gets something new. The fundamental theorem of coalgebras and comodules, which states

Theorem 4.2.2 (Fundamental theorem of coalgebras and comodules) *Every element in a coalgebra C is contained in a finite dimensional subcoalgebra. Any element of a C-comodule is contained in a finite dimensional subcomodule.*

tells us that general coalgebras behave a lot more like finite dimensional algebras than general infinite dimensional algebras do, and a correspondingly rich structure theory for the comodules has been developed, see Green [40] for example.

The category of coalgebras over a field has arbitrary coproducts. To see this, suppose $(C_\lambda, \Delta_\lambda, \epsilon_\lambda)_\lambda$ is an arbitrary family of coalgebras, and take the direct sum as vector spaces

$$C = \oplus_\lambda C_\lambda,$$

with canonical inclusion $i_\lambda : C_\lambda \to C$. Now consider the linear maps

$$\Delta_\lambda : C_\lambda \to C_\lambda \otimes C_\lambda \subset C \otimes C,\ \epsilon_\lambda : C_\lambda \to k.$$

By the universal property of coproducts, there exist unique maps

$$\Delta : C \to C \otimes C,\ \Delta \circ i_\lambda = \Delta_\lambda \tag{4.1}$$

$$\epsilon : C \to k,\ \epsilon \circ i_\lambda = \epsilon_\lambda. \tag{4.2}$$

That the coalgebra axioms are satisfied because all C_λ are coalgebras basically follows from the fact that

$$\mathrm{Hom}(C, A) = \mathrm{Hom}(\oplus_\lambda C_\lambda, A) \cong \prod_\lambda \mathrm{Hom}(C_\lambda, A),$$

and $\mathrm{Hom}(C, A)$ is an algebra with convolution product. The same kind of reasoning is valid for turning the coproduct of C-comodules into a C-comodule. Slightly more interesting, there is the following lemma

Lemma 4.2.3 *There is a functor*

$$\psi : \bigoplus_i \mathrm{CoMod}(C_i) \to \mathrm{CoMod}(C),$$

inducing an equivalence of categories.

Proof: Given C_i-comodules (W_i, δ_i), define $W = \oplus_i W_i$, and $\delta(w) = \oplus_i \delta_i(w_i)$, for $w = \oplus_i w_i$, and $w_i \in W_i$. This defines the functor on objects. The value on maps is equally obvious. An inverse functor can be constructed by noticing that for a C-comodule of the form $W = \bigoplus_i W_i$, such that each W_i is a C_i-comodule, one has the canonical projections

$$\pi_i : W \to W_i.$$

Suppose $w = \sum_i w_i$, then since

$$\pi_j(w) = w_j = \epsilon_j(w_{j[0]})w_{j[1]},$$

where $\delta_j(w_j) = w_{j[0]} \otimes w_{j[1]}$, the projection can also be obtained from w by

$$w \mapsto \delta(w) = \sum_i \delta_i(w_i) = \sum_i w_{i[0]} \otimes w_{i[1]} \mapsto w_{j[0]} \otimes w_{j[1]} \mapsto \epsilon_j(w_{j[0]})w_{j[1]}.$$

Motivated by this, for an arbitrary C-comodule (W, δ), define maps

$$p_i : W \to p_i(W) : w \mapsto \epsilon_i(w_{[0]})w_{[1]},$$

where $\delta(w) = w_{[0]} \otimes w_{[1]}$, and ϵ_i is extended to C by setting it equal to zero on C_j, $j \neq i$. It is routine to verify this satisfies our needs. □

This statement tells us that to understand the representation theory of C, it is sufficient to understand the representations of the C_i. This is applicable to the setting of universal coacting bialgebras since there is a decomposition of coalgebras

$$\underline{\mathrm{end}}(A) = \bigoplus_n \underline{\mathrm{end}}(A)_n.$$

The n-th graded part $\underline{\mathrm{end}}(A)_n$ is finite dimensional, so the general theory tells us that

$$\mathrm{CoMod}(\underline{\mathrm{end}}(A)_n) \cong \mathrm{Mod}(\underline{\mathrm{end}}(A)_n^*).$$

The right hand side of this isomorphism is the category of right modules. To make sure no mix-ups occur between left and right, we pass to the opposite algebra $(\underline{\mathrm{end}}(A)_n^*)^{op}$. Since each $\underline{\mathrm{end}}(A)_n^*$ is a finite dimensional algebra it's no surprise that quiver representations will show up.

Definition 4.2.4 *Define $Z_n(A)$ to be the subalgebra of the k-endomorphisms of $V^{\otimes n}$ that preserve the relation spaces. In other words*

$$Z_n(A) = \{\phi \in \mathrm{End}_k(V^{\otimes n}) \mid \forall 1 \leq i \leq n - N + 1 : \phi(R_i^n) \subset R_i^n\},$$

and if $n < N$, put $Z_n(A) = \mathrm{End}_k(V^{\otimes n})$.

Then, as expected, we have the following

Proposition 4.2.5 *As k-algebras, there is an isomorphism*

$$(\underline{\mathrm{end}}(A)_n^*)^{op} \cong Z_n(A)$$

Proof: In the case $n < N$, the isomorphism as vector spaces is obvious, since

$$\underline{\mathrm{end}}(A)_n = T(V^* \otimes V)_n = (V^* \otimes V)^{\otimes n},$$

and since V is finite dimensional, $V \otimes V^* \cong \mathrm{End}_k(V)$. To check that the multiplications coincide, it suffices to look at the basis of $T(V^* \otimes V)_n$, made up out of n-fold products of the z_i^j's. Using the shorthand $x_{\mathbf{i}} = x_{i_1} \cdots x_{i_n}$, for $1 \leq i_k \leq \dim(V)$, and the same for $\mathbf{j}$, $\mathbf{k}$ and $\mathbf{l}$, where x is some symbol, the meaning of which should be obvious from the context, we get for the comultiplication on $\underline{\mathrm{end}}(A)_n$:

$$\Delta(z_{\mathbf{i}}^{\mathbf{j}}) = \sum_{\mathbf{k}} z_{\mathbf{i}}^{\mathbf{k}} \otimes z_{\mathbf{k}}^{\mathbf{j}}.$$

The multiplication on $\underline{\mathrm{end}}(A)_n^*$ is then

$$z_{\mathbf{i}}^{\mathbf{k}*} \cdot z_{\mathbf{l}}^{\mathbf{j}*} = \delta_{\mathbf{l}}^{\mathbf{k}} z_{\mathbf{i}}^{\mathbf{j}*}.$$

As element of $\mathrm{End}_k(V^{\otimes n})$, we have

$$z_{\mathbf{i}}^{\mathbf{k}*} \mapsto \big(v \mapsto (x_{\mathbf{i}}^*)(v) \cdot x_{\mathbf{k}}\big),$$

so the multiplication is opposite to the one on $\mathrm{End}_k(V^{\otimes n})$, as we desire. In the case $n \geq N$, the vector space isomorphism is a little harder to see:

$$\underline{\mathrm{end}}(A)_n = (V^* \otimes V)^{\otimes n}/(\pi_N(R^\perp \otimes R))_n.$$

Since

$$(\pi_N(R^\perp \otimes R))_n = \sum_{i=0}^{n-N} (V^* \otimes V)^{\otimes i} \otimes \pi_N(R^\perp \otimes R) \otimes (V^* \otimes V)^{\otimes n-N-i},$$

and noticing that the terms in the sum are isomorphic to

$$Y_i = V^{*\otimes i} \otimes R^\perp \otimes V^{*\otimes n-N-i} \otimes V^{\otimes i} \otimes R \otimes V^{\otimes n-N-i},$$

the dual of $\underline{\mathrm{end}}(A)_n$ is exactly the orthogonal complement with respect to $\perp$ of the sum of the Y_i. In other words

$$\underline{\mathrm{end}}(A)_n^* \cong \left(\sum_{i=0}^{n-N} Y_i\right)^\perp = \bigcap_{i=0}^{n-N} Y_i^\perp,$$

which is exactly $Z_n(A)$, using $(V^* \otimes V)^* \cong \mathrm{End}_k(V)$ again, and the obvious equivalence induced by this isomorphism

$$\phi(Y_i) = 0 \Longleftrightarrow \phi(V^{\otimes i} \otimes R \otimes V^{\otimes n-N-i}) \subset V^{\otimes i} \otimes R \otimes V^{\otimes n-N-i},$$

where we identified $\phi \in \mathrm{End}_k(V^{\otimes n})$ and its image. The argument we made in the $n < N$ case concerning multiplication suffices for this case as well. $\square$

Using the grading on $\underline{\mathrm{end}}(A)$, there are vector space morphisms

$$m_{p,q} : \underline{\mathrm{end}}(A)_p \otimes \underline{\mathrm{end}}(A)_q \to \underline{\mathrm{end}}(A)_{p+q}.$$

Denoting $d = \{1, \ldots, \dim(V)\}$, we see that on the one hand

$$(\Delta \circ m_{p,q})(z_{\mathbf{i}}^{\mathbf{k}} \otimes z_{\mathbf{l}}^{\mathbf{j}}) = \sum_{\mathbf{s} \in d^{p+q}} z_{\mathbf{il}}^{\mathbf{s}} \otimes z_{\mathbf{s}}^{\mathbf{kj}},$$

while on the other

$$((m_{p,q} \circ m_{p,q}) \circ \Delta)(z_{\mathbf{i}}^{\mathbf{k}} \otimes z_{\mathbf{l}}^{\mathbf{j}}) = \sum_{\mathbf{r} \in d^p, \mathbf{t} \in d^q} z_{\mathbf{il}}^{\mathbf{rt}} \otimes z_{\mathbf{rt}}^{\mathbf{kj}},$$

where the Δ in the second equality is actually $(1 \otimes \tau \otimes 1) \circ (\Delta \otimes \Delta)$, and τ denotes the switch map. Since the two are obviously equal, $m_{p,q}$ is a coalgebra morphism, inducing an algebra morphism

$$\mu_{p,q} : Z_{p+q}(A) \to Z_p(A) \otimes Z_q(A).$$

These fit into the commutative diagram

$$\begin{tikzcd}
Z_{p+q}(A) \arrow[r, "\mu_{p,q}"] \arrow[d, hook] & Z_p(A) \otimes Z_q(A) \arrow[d, hook] \\
\mathrm{End}(V^{\otimes p+q}) \arrow[r, "\cong"] & \mathrm{End}(V^{\otimes p}) \otimes \mathrm{End}(V^{\otimes q})
\end{tikzcd}$$

and we see that the representation theory of $\underline{\mathrm{end}}(A)$ can be computed by studying $\mathrm{Mod}(Z_n(A))$. The monoidal structure on these modules can be computed by pulling back through $\mu_{p,q}$: given $M_1 \in \mathrm{Mod}(Z_p(A))$ and $M_2 \in \mathrm{Mod}(Z_q(A))$, the tensor product $M_1 \otimes M_2$ is a $Z_p(A) \otimes Z_q(A)$-module and the pullback gives a $Z_{p+q}(A)$-module. The following immediate corollary provides us with an ample amount of representations, and illustrates the importance of the lattices considered in the previous chapter.

Corollary 4.2.6 *Set $\mathcal{L}_n(A)$ to be the lattice of subspaces of $V^{\otimes n}$ generated by the n-th relation spaces $(R_i^n)_i$. All objects in this lattice are $Z_n(A)$-modules and thus $\underline{\mathrm{end}}(A)$-comodules.*

4.3 Representation theory in the distributive setting

To prove and formulate the main theorem in this section on the representations of $\underline{\mathrm{end}}(A)$, for A distributive, we will need to consider localization in abelian categories. So as not to confuse the reader once we start talking about distributive algebras, we describe this construction now. Anyone familiar with localization in the setting of non-commutative ring theory can safely skip the following section, without really missing out on anything.

Pinpointing abstract nonsense

Throughout this interlude, we will be rather informal and not consider all details necessary to define the right concepts. The material is taken from the Stacks project [20], and this section is only meant to clarify the statement of Theorem 4.3.13.

Definition 4.3.1 *Let $\mathcal{C}$ be any category. A set of arrows S in $\mathcal{C}$ is called a multiplicative set if the following conditions hold:*

1. *S contains the identity morphisms of $\mathcal{C}$, and S is closed under composition (whenever this is defined)*

2. *Every solid diagram of the form*

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "t"] & Y \arrow[d, dashed, "s"] \\ Z \arrow[r, dashed, "f"] & W \end{tikzcd}$$

   with $t \in S$ can be completed to a commutative dotted diagram with $s \in S$ (+ the corresponding 'right' condition)

3. *For two morphisms $f, g : X \to Y$ and $t \in S$ with target X such that $f \circ t = g \circ t$, there exists an $s \in S$ with source X such that $s \circ f = s \circ g$ (+ the corresponding 'right' condition)*

For a category $\mathcal{C}$ and a multiplicative system S, one can define a category $S^{-1}\mathcal{C}$ as follows.

- The objects are just the objects of $\mathcal{C}$
- Morphisms $X \to Y$ are given by pairs $(f : X \to Y', s : Y \to Y')$, where $s \in S$, up to equivalence; we will think of such a morphism as $s^{-1}f : X \to Y$
- Two morphisms $(f_1 : X \to Y_1, s_1 : Y \to Y_1), (f_2 : X \to Y_2, s_2 : Y \to Y_2)$ are said to be equivalent if there exists another morphism $(f_3 : X \to Y_3, s_3 : Y \to Y_3)$ and morphisms $u : Y_1 \to Y_3$ and $v : Y_2 \to Y_3$ in $\mathcal{C}$ such that the following diagram commutes

$$\begin{tikzcd} & Y_1 \arrow[d, "u"] & \\ X \arrow[ur, "f_1"] \arrow[r, "f_3"] \arrow[dr, "f_2"'] & Y_3 & Y \arrow[ul, "s_1"'] \arrow[l, "s_3"] \arrow[dl, "s_2"] \\ & Y_2 \arrow[u, "v"] & \end{tikzcd}$$

- The composition of $(f : X \to Y', s : Y \to Y')$ and $(g : Y \to Z', t : Z \to Z')$ is defined to be the equivalence class of a pair $(h \circ f : X \to Z'', u \circ t : Z \to Z'')$, where $h, u \in S$ are chosen such that

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "s"] & Z' \arrow[d, "u"] \\ Y' \arrow[r, "h"] & Z'' \end{tikzcd}$$

  commutes

This turns out to be nicely defined (a technical though not very hard check). There is also a right version, which turns out to be canonically isomorphic to $S^{-1}\mathcal{C}$. This localization then satisfies the following universal property.

Lemma 4.3.2 *Let $\mathcal{C}$ be a category and S a multiplicative system. Then*

- *There is a functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$, which is the identity on objects and sends a morphism $X \to Y$ to the morphism $(X \to Y, id_Y)$.*
- *For any $s \in S$ the morphism $Q(s)$ is an isomorphism in $S^{-1}\mathcal{C}$*
- *If $G : \mathcal{C} \to \mathcal{D}$ is any functor such that $G(s)$ is invertible for any $s \in S$, then there is a unique functor $H : S^{-1}\mathcal{C} \to \mathcal{D}$ such that $H \circ Q = G$*

The functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ is called the localization functor. In the case of an abelian category $\mathcal{A}$, the localization has even nicer properties; in fact, it can be proven that $S^{-1}\mathcal{A}$ is an abelian category, and the localization functor is an exact functor.

Definition 4.3.3 *A Serre subcategory $\mathcal{C}$ of an abelian category $\mathcal{A}$ is a non-empty full subcategory such that for every exact sequence in $\mathcal{A}$*

$$0 \to A \to B \to C \to 0,$$

A and C are objects in $\mathcal{C}$ iff B is an object in $\mathcal{C}$.

Definition 4.3.4 *Given two abelian categories $\mathcal{A}, \mathcal{B}$, and an exact functor $F : \mathcal{A} \to \mathcal{B}$. The full subcategory of objects C in $\mathcal{A}$ such that $F(C) = 0$ is called the kernel of F, which we denote by Ker(F). This is always a Serre subcategory.*

Theorem 4.3.5 *Given an abelian category $\mathcal{A}$ and a Serre subcategory $\mathcal{C}$, there exists an abelian category $\mathcal{A}/\mathcal{C}$ and an exact functor*

$$F : \mathcal{A} \to \mathcal{A}/\mathcal{C},$$

which is essentially surjective and such that Ker(F) is exactly $\mathcal{C}$. The duo $\mathcal{A}/\mathcal{C}$ and F then satisfy the following universal property: for any exact functor $G : \mathcal{A} \to \mathcal{B}$ such that $\mathcal{C} \subset \mathrm{Ker}(G)$, there exists a unique exact functor $H : \mathcal{A}/\mathcal{C} \to \mathcal{B}$ such that the following diagram commutes

$$\begin{tikzcd} \mathcal{A} \arrow[r, "G"] \arrow[dr, "F"'] & \mathcal{B} \\ & \mathcal{A}/\mathcal{C} \arrow[u, dashed, "H"'] \end{tikzcd}$$

Proof:[sketch] Take

$$S = \{f \in \mathrm{Arrows}(\mathcal{A}) \mid \mathrm{Ker}(f), \mathrm{Coker}(f) \in \mathcal{C}\};$$

this set is then a multiplicative system. To see that Condition 1 is fulfilled, it suffices to observe that there are exact sequences

$$0 \to \mathrm{Ker}(f) \to \mathrm{Ker}(gf) \to \mathrm{Ker}(g) \to 0$$

$$0 \to \mathrm{Coker}(f) \to \mathrm{Coker}(gf) \to \mathrm{Coker}(g) \to 0,$$

and use the defining property of Serre subcategories. To check Condition 2, consider a diagram of the form

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "t"] & B \arrow[d, dashed, "s"] \\ C \arrow[r, dashed, "f"] & C \coprod_A B \end{tikzcd}$$

Set W to be the corkernel of the map

$$(g, -f) : A \to C \oplus B$$

Then $\mathrm{Ker}(t) \to \mathrm{Ker}(s)$ is surjective, and $\mathrm{Coker}(t) \to \mathrm{Coker}(s)$ is an isomorphism, showing that $s \in S$. For Condition 3, consider morphisms $f, g : B \to C$ and a morphism $s : A \to B$ in S such that $f \circ s = g \circ s$. Then $I = \mathrm{Im}(f - g) \subset C$ is a quotient of the cokernel of s, and thus belongs to $\mathcal{C}$. This means that $t : C \to C/I$ is an element in S such that $t \circ (f - g) = 0$. The right properties have proofs exactly dual to these. This allows us to define $\mathcal{A}/\mathcal{C}$ as $S^{-1}\mathcal{A}$, and put F to be the localization functor. If X is an element in $\mathrm{Ker}(F)$, then it can be shown that $0 : X \to Z$ is an element of S, which means that X is an object in $\mathcal{C}$, and we get that $\mathrm{Ker}(F) = \mathcal{C}$. If G is an exact functor as in the statement of the proposition, then G turns every element of S into an isomorphism. By the universal property of localization there is a functor $H : \mathcal{A}/\mathcal{C} \to \mathcal{B}$ satsifying the claim. $\square$

Corollary 4.3.6 *If $\mathcal{A}, \mathcal{B}$ are abelian categories and $F : \mathcal{A} \to \mathcal{B}$ is an exact functor, then there is an induced functor*

$$\overline{F} : \mathcal{A}/\mathrm{Ker}(F) \to \mathcal{B}$$

which is faithful.

The situation we will encounter later on resembles this but is not completely the same.

Definition 4.3.7 *A Serre subcategory $\mathcal{C}$ of an abelian category $\mathcal{A}$ is called a localizing subcategory if the morphisms which are invertible in the quotient category $\mathcal{A}/\mathcal{C}$ are precisely the images of morphisms in S (like we defined it in the proof of the previous theorem).*

More than one Serre subcategory can give rise to the same quotient category, and this ambiguity is resolved by considering the corresponding localizing subcategory. Given a Serre subcategory $\mathcal{C}$ of $\mathcal{A}$, it is a localizing subcategory if and only if Q has a right adjoint, and in that case $\mathcal{A}/\mathcal{C}$ is Grothendieck if and only if $\mathcal{A}$ is. The Gabriel-Popescu theorem says that any Grothendieck category can be obtained as $\mathrm{Mod}(R)/\mathcal{T}$, where $\mathcal{T}$ is one of the localizing subcategories. For our purposes, it can be proven that a Serre subcategory is a localizing subcategory if and only if it is closed under arbitrary direct sums, see [20].

Classifying all comodules

From now on, we assume $A = TV/(R)$ is a distributive algebra with N-ary relations. Surprisingly, the category $\mathrm{Cube}_\bullet$ makes an appearance again, providing all representations of such an A. To get there however, we first need

Definition 4.3.8 *For $I \subset \{1, \ldots, n-N+1\}$, and*

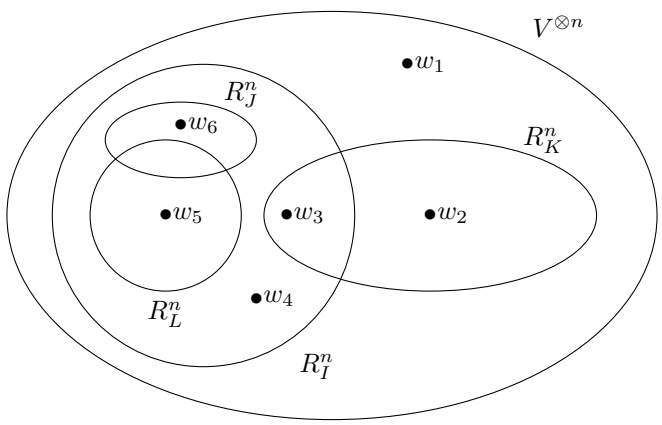
$$C_I^n = R_I^n / \sum_{J \supsetneq I} R_J^n,$$

the subset I is said to be admissible if $C_I \neq 0$. If $n < N$, $I = \emptyset$ and $R_\emptyset = C_\emptyset = V^{\otimes n}$. Denote by Adm *the set of admissible subsets.*

This definition allows us to determine all $Z_n(A)$-representations using some elementary non-commutative algebra.

Theorem 4.3.9 *The indecomposable projective $Z_n(A)$-representations are exactly the R_I^n, for I admissible.*

Proof: By Proposition 3.1.3 and the fact that A is distributive, there is a basis $(w_\alpha)_\alpha$ for $V^{\otimes n}$ such that each vector space in $\mathcal{L}_n(A)$ is spanned by some subset of this basis. Each of the subspaces C_I^n are then spanned by those w_α which don't belong to an R_J^n, for $J \supsetneq I$. If we look at ever bigger subsets of $\{1, \ldots, n-N+1\}$ containing I, and take the w_α satisfying the above condition, we will eventually get all of R_I^n. For example, in the drawing below



from C_I^n we get w_4, from $C_{I\cap K}^n$ we get w_3, C_L^n gives w_5, and C_J^n provides w_6. In conclusion,

$$R_I^n = \bigoplus_{J \supset I} C_J^n.$$

The basis $(w_\alpha)_\alpha$ also allows for a decomposition of $\mathrm{End}(V^{\otimes n})$ as

$$\mathrm{End}(V^{\otimes n}) = \bigoplus_I \mathrm{Hom}(C_I^n, V^{\otimes n}) = \bigoplus_{I,J} \mathrm{Hom}(C_I^n, C_J^n).$$

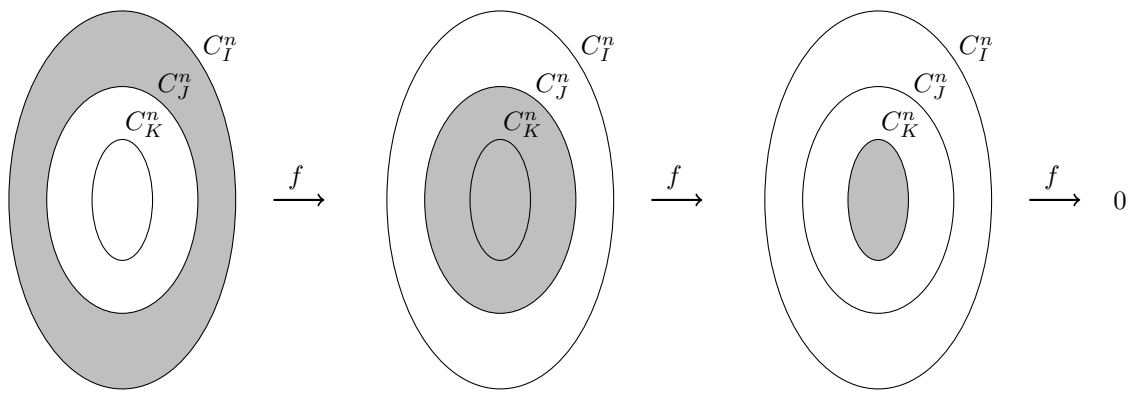
To see the first equality, it suffices to notice that every basis element w_α is contained in one of the C_I^n, and essentially the same for the second equality. Set $e_\alpha \in \mathrm{End}(V^{\otimes n})$ to be the projection on kw_α. In fancier terms, this is a primitive idempotent. By the special property of our basis, $e_\alpha(R_i^n) \subset R_i^n$, hence $e_\alpha \in Z_n(A)$. Taking all e_α's we thus get a maximal set of primitive, orthogonal idempotents for $Z_n(A)$. Since the elements of $Z_n(A)$ map relation spaces into themselves, the picture tells us there is a decomposition

$$Z_n(A) = \bigoplus_{J \supset I} \mathrm{Hom}(C_I^n, C_J^n).$$

Remember that the Jacobson radical of a general ring R, which we denote by $\mathrm{rad}(R)$, is the intersection of all maximal left ideals, and measures the obstruction to being semisimple (and thus being subject to the Wedderburn-Artin theorem). In the case of a finite dimensional algebra, or more generally an Artinian ring, it is easily seen that this definition is equivalent to demanding $\mathrm{rad}(R)$ is the largest nilpotent two-sided ideal. Rewrite $Z_n(A)$ as

$$Z_n(A) = \bigoplus_{J \supsetneq I} \mathrm{Hom}(C_I^n, C_J^n) \oplus \bigoplus_{I \notin \mathrm{Adm}} \mathrm{Hom}(C_I^n, C_I^n) \oplus \bigoplus_{I \in \mathrm{Adm}} \mathrm{Hom}(C_I^n, C_I^n).$$

Then the first and second big sums will belong to $\mathrm{rad}(Z_n(A))$, since some power will have to be zero in the first case, as per the following drawings



In the second case, C_I^n itself is zero so obviously $\mathrm{Hom}(C_I^n, C_I^n) = 0$. Thus

$$Z_n(A)/\mathrm{rad}(Z_n(A)) = \bigoplus_{I \in \mathrm{Adm}} \mathrm{Hom}(C_I^n, C_I^n).$$

Since the identity on $Z_n(A)$ can be written as sum of the e_α, and the e_α are non-central, orthogonal idempotents, the general theory of finite dimensional algebras tells us that each $Z_n(A)e_\alpha$ is an indecomposable projective $Z_n(A)$-module. Again, by the general theory, two of these are isomorphic if and only if they become isomorphic in the semisimple quotient, which we just calculated. This tells us that by picking an α_I such that $w_{\alpha_I} \in C_I^n$, for I admissible, the indecomposable projectives of $Z_n(A)$ are exactly given by the

$$Q_I = Z_n(A)e_{\alpha_I}.$$

Using our precious calculations again, one sees that

$$Q_I = \bigoplus_{J \supset I} \mathrm{Hom}(kw_{\alpha_I}, C_J^n) = \mathrm{Hom}(kw_{\alpha_I}, \bigoplus_{J \supset I} C_J^n) = \mathrm{Hom}(kw_{\alpha_I}, R_I^n) \cong R_I^n,$$

proving the proposition. □

Corollary 4.3.10 *For I, J admissible:*

$$\mathrm{Hom}_{Z_n(A)}(R_J^n, R_I^n) = \begin{cases} k & \text{if } J \supset I \\ 0 & \text{otherwise} \end{cases}$$

In the first case, the single generator is given by the inclusion $R_J^n \subset R_I^n$.

Proof: It is easy to see the following map is an isomorphism of vector spaces

$$\mathrm{Hom}_{Z_n(A)}(Z_n(A)e_{\alpha_J}, Z_n(A)e_{\alpha_I}) \to e_{\alpha_J} Z_n(A) e_{\alpha_I} : f \mapsto f(e_{\alpha_J}).$$

Using the isomorphism in the last step of the previous theorem, it is sufficient to prove

$$e_{\alpha_J} Z_n(A) e_{\alpha_I} = \begin{cases} k & \text{if } J \supset I \\ 0 & \text{otherwise} \end{cases}$$

This follows immediately by the definition of the e_α. □

Corollary 4.3.11 *The simple $Z_n(A)$-representations and their projective covers are given respectively by the C_I^n, for I admissible, and the corresponding R_I^n.*

Proof: The simple top of the projective $R_I^n \cong Q_I$ can be computed as the cokernel of the map

$$\bigoplus_{Q_J \not\cong Q_I} Q_J \otimes \mathrm{Hom}_{Z_n(A)}(Q_J, Q_I) \to Q_I.$$

Using the theorem and the previous corollary, this is the same as the cokernel of

$$\bigoplus_{R_J^n \subsetneq R_I^n} R_J^n \to R_I^n,$$

and by the definition of C_I^n, this is exactly what we have to prove. □

The most important corollary of all this, is that the representations of $Z_n(A)$ can be described as representations of a specific quiver with relations.

Corollary 4.3.12 *The category* $\mathrm{Mod}(Z_n(A))$ *is equivalent to a category* $\mathrm{Rep}(Q_{A,n-N+1})$ *of quiver representations, where $Q_{A,n-N+1}$ is the following quiver with relations:*

- $Q_0 = \{x_I \mid I \subset \{1, \cdots, n-N+1\}, I \text{ admissible}\}$
- $Q_1 = \{x_{IJ} : x_I \to x_J \mid I \subset J\}$
- $\rho = (x_{JK} x_{IJ} = x_{IK} \mid I \subset J \subset K)$

In the degenerate case $n \leq N-1$, $Q_{A,n-N+1}$ is the quiver consisting of a single vertex $x_\emptyset$ and no arrows.

Proof: This is essentially a reformulation of Theorem 4.3.9 and Corollary 4.3.10 above. Explicitly, take the exact functor

$$\mathrm{Hom}_{Z_n(A)}(\oplus_{J \in \mathrm{Adm}} R_J, -) : \mathrm{Mod}(Z_n(A)) \to \mathrm{Rep}(Q_{A,n-N+1}).$$

This will take an indecomposable projective R_I^n for I admissible, to the representation of $Q_{A,n-N+1}$ that has k at every vertex x_J, $J \supset I$, and zero elsewhere; this is exactly the indecomposable projective P_I^{n-N+1} corresponding to the vertex x_I. Since the morphisms between projective indecomposables in $\mathrm{Mod}(Z_n(A))$ are given by inclusions, the functor also goes well on morphisms. Every (fd) projective module is a direct sum of projective indecomposables, and every module has a projective cover (we are in the semiperfect

setting), so the images of the R_I^n even characterize the functor. The fact that it is an equivalence follows by considering the inverse functor

$$\mathcal{F}^{\circ}_{A,n} : \mathrm{Rep}(Q_{A,n-N+1}) \to \mathrm{Mod}(Z_n(A)),$$

which sends the projective quiver representation P_I^{n-N+1} to R_I^n. $\square$

All of this allows for the application of methods developed in the previous chapter, on the basis of Lemma 3.3.1. Define a functor

$$\mathcal{F}_{A,n} : \mathrm{Cube}_{n-N+1} \to \mathrm{CoMod}(\underline{\mathrm{end}}(A)_n),$$

by sending the projective indecomposables P_I^{n-N+1} to R_I^n, and the obvious for morphisms. Demanding this functor to be exact completely determines it: again, this is basically because we are working in a Krull-Schmidt setting. Doing this for all n and summing over n, we get an exact monoidal functor

$$\mathcal{F}_A : \mathrm{Cube}_\bullet \to \mathrm{CoMod}(\underline{\mathrm{end}}(A))$$

This functor extends the functor F_A we defined in Section 3.3, or in more formal terms, it is an enhancement of F_A, so we have a commutative triangle

$$\begin{tikzcd}
 & \mathrm{CoMod}(\underline{\mathrm{end}}(A)_n) \arrow[dr, "U"] & \\
\mathrm{Cube}_{n-N+1} \arrow[ur, "\mathcal{F}_{A,n}"] \arrow[rr, "F_{A,n}"] & & \mathrm{Vect}
\end{tikzcd}$$

where U is the forgetful functor. We now come to the main theorem, completely describing the representations of $\underline{\mathrm{end}}(A)$ in terms of a quotient of the category $\mathrm{Cube}_\bullet$.

Theorem 4.3.13 *The functor $\mathcal{F}_A$ defines an equivalence of categories*

$$\overline{\mathcal{F}_A} : \mathrm{Cube}_\bullet / \mathcal{S}_A \to \mathrm{CoMod}(\underline{\mathrm{end}}(A)),$$

where $\mathcal{S}_A$ is the localizing subcategory of $\mathrm{Cube}_\bullet$ *which is generated by those simples S_I^l for which I is not admissible.*

Proof: Consider the diagram

$$\begin{tikzcd}
 & \mathrm{Rep}(Q_{A,n-N+1}) \arrow[dl, bend left, "\mathrm{Ind}"] \arrow[dr, bend left, "\mathcal{F}^{\circ}_{A,n}", "\cong"'] & \\
\mathrm{Cube}_{n-N+1} \arrow[ur, bend left, "\mathrm{Res}"] \arrow[rr, bend right, "\mathcal{F}_{A,n}"'] & & \mathrm{Mod}(Z_n(A))
\end{tikzcd}$$

where Res is the restriction functor

$$\mathrm{Res} : \mathrm{Rep}(Q_{n-N+1}) \to \mathrm{Rep}(Q_{A,n-N+1}),$$

and remembering that Q_n was the quiver defined just above Definition 3.2.6, and $\mathrm{Cube}_{n-N+1} = \mathrm{Rep}(Q_{n-N+1})$. The functor

$$\mathrm{Ind} : \mathrm{Rep}(Q_{A,n-N+1}) \to \mathrm{Rep}(Q_{n-N+1})$$

is defined as follows: since the path algebra of $Q_{A,n-N+1}$ is a subalgebra of the path algebra of Q_{n-N+1}, a module M for the former can be induced to a module for the latter by defining

$$\mathrm{Ind}(M) = kQ_{n-N+1} \otimes_{kQ_{A,n-N+1}} M.$$

It is well known that given a ring map, the induction functor is left adjoint to the restriction functor. The idea behind this adjunction is that for a ring morphism $f : A \to B$, an A-module M, a B-module N and a map of B-modules $b : B \otimes_A M \to N$, we can define a map of A-modules $a : M \to {}_AN$ by $a(m) = b(1 \otimes m)$, and conversely, given a, define b by $b(x \otimes m) = x \cdot a(m)$.

We prove that the diagram is commutative in both possible ways

1. $\mathcal{F}^\circ_{A,n} = \mathcal{F}_{A,n} \circ \mathrm{Ind}$: since Ind is just tensoring with a module, it's right exact and commutes with direct sums, as do $\mathcal{F}^\circ_{A,n}$ and $\mathcal{F}_{A,n}$. To show the equality then, it is sufficient to check projectives. For an admissible I we have

$$(\mathcal{F}_{A,n} \circ \mathrm{Ind})(P_I^{n-N+1}) = \mathcal{F}_{A,n}(P_I) = R_I = \mathcal{F}^\circ_{A,n}(P_I)$$

2. $\mathcal{F}_{A,n} = \mathcal{F}^\circ_{A,n} \circ \mathrm{Res}$: suppose $M \in \mathrm{Cube}_{n-N+1}$. From the previous, we know that

$$(\mathcal{F}^\circ_A \circ \mathrm{Res})(M) = \mathcal{F}_A(\mathrm{Ind} \circ \mathrm{Res}(M)).$$

   The counit-unit adjunction provides the obvious map

$$\epsilon_M : \mathrm{Ind} \circ \mathrm{Res}(M) \to M : x \otimes m \to x \cdot m.$$

   This map is surjective, since $m = \epsilon_M(1 \otimes m)$. Applying the exact functor $\mathcal{F}_A$, and noticing that the kernel of the map corresponds to the non-admissible simples, we find that

$$\mathcal{F}_{A,n}(\mathrm{Ind} \circ \mathrm{Res}(M)) = \mathcal{F}_{A,n}(M).$$

Now put $\mathcal{S}_{A,n-N+1} = \mathcal{S}_A \cap \mathrm{Cube}_{n-N+1}$. This localizing subcategory of Cube_{n-N+1} then belongs to $\mathrm{Ker}(\mathrm{Res}) \subset \mathrm{Ker}(\mathcal{F}_A)$. It is obvious that

$$\overline{\mathrm{Res}} : \mathrm{Cube}_{n-N+1}/\mathcal{S}_{A,n-N+1} \to \mathrm{Rep}(Q_{A,n-N+1})$$

is an equivalence of categories. This gives us a commutative diagram

$$\begin{tikzcd}
 & \mathrm{Rep}(Q_{A,n-N+1}) \arrow[dr, dashed, "\mathcal{F}^\circ_{A,n}", "\cong"'] & \\
\mathrm{Cube}_{n-N+1}/\mathcal{S}_{A,n-N+1} \arrow[ur, "\mathrm{Res}", "\cong"'] \arrow[rr, "\overline{\mathcal{F}_{A,n}}"'] & & \mathrm{Mod}(Z_n(A))
\end{tikzcd}$$

which proves that $\overline{\mathcal{F}_{A,n}}$ is an equivalence and since everything commutes with direct sums, also $\overline{\mathcal{F}_A}$ is an equivalence. $\square$

Corollary 4.3.14 *If A also satisfies the extra condition, then the representation category* $\mathrm{CoMod}(\underline{\mathrm{end}}(A))$ *is completely determined by the numbers* $(\dim(J_n))_n$*, where as before*

$$J_n = R_1^n \cap \cdots \cap R_{n-N+1}^n.$$

Proof: From the previous theorem it suffices to know which of the simples S_I^l get mapped to zero by $\mathcal{F}_A$. Fixing an l, we can look at F_A. Like before, a projective resolution of S_I^l is given by

$$\cdots \to \bigoplus_{\substack{J \supset I \\ |J-I|=2}} P_J^l \to \bigoplus_{\substack{J \supset I \\ |J-I|=1}} P_J^l \to S_I^l \to 0,$$

and using the formula for the Euler characteristic of the sequence, it is sufficient to know the dimensions of $F_A P_I^l$. If I has a hole of size $\leq N-2$, delimited by $i < j$, then there exists a k in between such that $F_A P_I^l = F_A P_{I \cup \{k\}}^l$, because of the extra condition. So we can safely assume that I has no holes of size $\leq N-2$. If I contains a hole of size $> N-2$, then P_I^l can be rewritten as

$$P_I^l = P_{I_1}^{l_1} \otimes P_{I_2}^{l_2},$$

for proper values of l_1, l_2, I_1 and I_2, by definition of $\otimes$. The same holds, if I does not contain 1 or l. The only case that remains is if I has no holes, meaning $I = \{1, \ldots, l\}$. In this case, $F_A P_I^l = J_l$. $\square$

Corollary 4.3.15 *If A is Koszul, then* $\mathrm{CoMod}(\underline{\mathrm{end}}(A))$ *is completely determined by the Hilbert series of A.*

Proof: For general N, the degree m part of the Koszul complex looks like

$$\cdots \to A_{m-2N} \otimes J_{2N} \xrightarrow{\delta_{m-2N}^{N-1}} A_{m-N-1} \otimes J_{N+1} \xrightarrow{\delta_{m-N-1}} A_{m-N} \otimes R \xrightarrow{\delta_{m-N}^{N-1}} A_{m-1} \otimes V \xrightarrow{\delta_{m-1}} A_m \to 0,$$

and the Hilbert series is obviously not enough to recover all J_n. In the case $N = 2$ however, this sequence reduces to

$$\cdots \to A_{m-4} \otimes J_4 \xrightarrow{\delta_{m-4}} A_{m-3} \otimes J_3 \xrightarrow{\delta_{m-3}} A_{m-2} \otimes R \xrightarrow{\delta_{m-2}} A_{m-1} \otimes V \xrightarrow{\delta_{m-1}} A_m \to 0,$$

and one can inductively recover the dimensions of all J_n from these exact sequences for different values of m, and the dimensions of the A_k. $\square$

5

A non-commutative coordinate ring for $M(n,k)$

The material of previous chapters will be applied to study the simplest affine variety, affine n-space $\mathbb{A}^n$, with coordinate ring $A = k[x_1, \ldots, x_n]$. The bialgebra $\underline{\mathrm{end}}(A)$ will be interpreted as a non-commutative version of the coordinate ring of the monoid of $n \times n$-matrices. To justify this interpretation, we will provide a link with the representation theory of this monoid, via the theory of Young diagrams. Since the final chapter will build upon this, we give an introduction to the representation theory of the symmetric and general linear groups, including Schur-Weyl duality, and symmetric function theory.

5.1 Generalities on algebraic groups

Consider the algebraic monoid $M_n = M(n,k)$ and the algebraic group $GL_n = GL(n,k)$ (by an algebraic group or monoid we mean a linear algebraic group or monoid over an algebraically closed field k of characteristic 0). As an algebraic group, GL_n is reductive.

Definition 5.1.1 *The radical of an algebraic group G is the maximal, connected, solvable normal subgroup. The unipotent radical of G is the maximal normal unipotent subgroup, which is also the unipotent part of the radical. A subgroup H is unipotent if every element $h \in H \subset G \subset GL_n$ is unipotent, i.e. $(h-1)^m = 0$, for some m. A linear algebraic group is reductive if the unipotent radical is trivial.*

For GL_n, the radical is given by the scalar matrices. The only non-trivial thing to show is the maximality, which can be deduced from the exact sequence

$$1 \to k^* \to GL_n \to PGL_n \to 1,$$

and remembering that PGL_n is simple. The set of unipotent elements in k^* is obviously trivial, showing that GL_n is reductive. In the case of algebraic groups, one is chiefly concerned with polynomial representations ρ, for which the entries of a representing matrix $\rho(g)$ are polynomials in the entries of the matrix $g \in GL_n$. As algebraic variety, GL_n is the open set in affine n^2-dimensional space where the polynomial $\det(g)$ doesn't vanish. The regular functions on GL_n are therefore generated by polynomials in the entries of g and the

multiplicative inverse of the determinant. If the entries of $\rho(g)$ are regular functions on GL_n, we say that ρ is a rational representation. Since the determinant function is itself a 1-dimensional representation of GL_n, any rational representation ρ gives rise to another by $(\det)^n \otimes \rho$, for any integer n. Since regular functions on GL_n have denominator a power of the determinant, every rational representation can be turned into a polynomial one by tensoring with a high enough power of the determinant representation. It is for this reason that the theories of rational and polynomial representations of algebraic groups are essentially interchangeable. If the representation is not finite dimensional, it is called rational if every cyclic submodule is finite dimensional (and hence all finitely generated submodules are), and every finite dimensional submodule is rational. Submodules, homomorphic images, sums and tensor products of rational modules are all rational.

For an algebraic group G, the coordinate algebra is denoted $\mathcal{O}(G)$, and is obviously a commutative Hopf algebra. In the case of GL_n, we have $\mathcal{O}(GL_n) = k[c_{11}, \ldots, c_{nn}, 1/det]$. Given a rational (right) representation V of G, with basis $(x_i)_i$, and coefficient functions $f_{ij} \in \mathcal{O}(G)$, defined by

$$x_i g = \sum_{j \in I} x_j f_{ij}(g),$$

there is on V also the structure of a (left) $\mathcal{O}(G)$-comodule, via the map

$$\tau : V \to k[G] \otimes V : x_i \mapsto \sum_{j \in I} f_{ij} \otimes x_j.$$

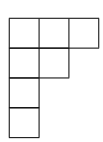
This definition is easily seen to be independent of the choice of basis and this gives an equivalence of categories between rational G-modules and $\mathcal{O}(G)$-comodules. One proves that for reductive algebraic groups, rational representations are completely reducible. It should be noted that the equivalence reductive $\Leftrightarrow$ completely reducible only holds in characteristic 0. In prime characteristic, complete reducibility is almost never the case. These and other basic facts on algebraic groups can be found in Humphreys [46].

5.2 Representation theory of the symmetric and general linear groups

In this section we will give the main ideas behind the use of Young diagrams in the representation theory of GL_n, Schur-Weyl duality, and the connection to symmetric functions. This theory is classic, and usually proofs are omitted, though our way of presenting these results is not standard. A detailed treatment can be found in the textbook by Fulton and Harris [34].

Representations of S_n

A Young diagram is just a collection of boxes, like so



and denoted by $\lambda = (\lambda_1, \ldots, \lambda_k)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ are positive integers that denote the row length. This is a partition of the total number of boxes, say n. A Young tableau is a Young diagram, such that each

box contains a positive integer, that is weakly increasing across each row, and strictly increasing along the columns. A standard tableau is a Young tableau with entries belonging to $\{1, \ldots, n\}$, each occurring once. For example,

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 3 & 3 \\ \cline{1-2} 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} 6 \\ \cline{1-1} 7 \\ \cline{1-1}\end{array}$$

are a (non-standard) Young tableau and a standard tableau. The conjugate μ of a Young diagram λ is the Young diagram obtained by reflecting along the skew diagonal. In our running example

$$\begin{array}{|c|c|c|c|}\hline & & & \\ \hline & \\ \cline{1-2} \\ \cline{1-1}\end{array}$$

Denote by T, T' numberings of a Young diagram $\lambda = (\lambda_1, \ldots, \lambda_k)$ (with n boxes) by the entries $1, \ldots, n$ with no repeats allowed. Then there is an obvious action of S_n on the numberings; for example

$$(12)(56) \cdot \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} 6 \\ \cline{1-1} 7 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|c|}\hline 2 & 3 & 6 \\ \hline 1 & 4 \\ \cline{1-2} 5 \\ \cline{1-1} 7 \\ \cline{1-1}\end{array}$$

For a fixed numbering T, denote by $R(T)$ the subgroup of S_n which permutes the rows amongst themselves. This is called the row group, and is isomorphic to a product of symmetric groups $S_{\lambda_1} \times \cdots \times S_{\lambda_k}$. In this form it is often called a Young subgroup. The analogous definition can be made for the columns, obtaining a subgroup $C(T)$, the column group. Notice that these subgroups are compatible with the S_n-action:

$$R(\sigma \cdot T) = \sigma \cdot R(T) \cdot \sigma^{-1}, \text{ and } C(\sigma \cdot T) = \sigma \cdot C(T) \cdot \sigma^{-1}$$

A tabloid is an equivalence class of numberings (with distinct numbers $1, \ldots, n$) of λ, where

$$T \sim T' \Leftrightarrow \text{ the corresponding rows contain the same entries} \Leftrightarrow T' = p \cdot T, \text{ for some } p \in R(T)$$

The class of T will be denoted $\{T\}$. Then the S_n-action is seen to carry over to tabloids, and the orbit of $\{T\}$ under this action is exactly $S_n/R(T)$. The Young symmetrizers associated to a numbering T are defined to be

$$\begin{aligned} a_T &= \sum_{p \in R(T)} p \\ b_T &= \sum_{q \in C(T)} \operatorname{sgn}(q) q \\ c_T &= b_T \cdot a_T \end{aligned}$$

where all operations take place in the group ring $\mathbb{C}S_n$. Denote by M^λ the complex vector space with basis

$$\{T\}, T \text{ of shape } \lambda, \lambda \text{ a partition of } n$$

Then M^λ is a left $\mathbb{C}S_n$-module. To a numbering T of λ, we associate the vector $v_T \in M^\lambda$ by

$$v_T = b_T \cdot \{T\}.$$

One can then define the Specht module S^λ as the subspace of M^λ generated by

$$v_T, T \text{ a numbering of } \lambda$$

This can be shown to be a $\mathbb{C}S_n$-module, and for any numbering T of λ, one has $\mathbb{C}S_n \cdot v_T = S^\lambda$. These completely determine the (complex) representation theory of S_n.

Theorem 5.2.1 *For each partition λ of n, S^λ is an irreducible representation of S_n. Moreover, every irreducible representation of S_n is isomorphic to exactly one S^λ. A basis of S^λ is given by the v_T, where T is a standard tableau on λ.*

Proof: See Fulton and Harris [34], Lecture 4, §4.2. □

Dually, one can introduce column tabloids as equivalence classes of numberings of a Young diagram, under the relation

$$T \sim T' \Leftrightarrow \text{ the corresponding columns contain the same entries}$$

This time however, we also introduce an orientation: $+$ or $-$. Two equivalent numberings have the same or opposite orientation depending on whether the permutation taking one to the other is even or odd. For example

$$\begin{array}{|c|c|c|}\hline 4 & 2 & 5 \\ \hline 1 & 3 \\ \cline{1-2} 7 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array} = -\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 4 & 3 \\ \cline{1-2} 7 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 4 & 2 \\ \cline{1-2} 7 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array} = -\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 4 & 2 \\ \cline{1-2} 6 \\ \cline{1-1} 7 \\ \cline{1-1}\end{array}$$

An oriented column tabloid defined by a numbering T is denoted by $[T]$, and $-[T]$ is used for that defined by an odd permutation preserving the columns. Introduce $\widetilde{M}^\lambda$ to be the $\mathbb{C}$-vector space with basis

$$[T],\ T \text{ is a numbering of } \lambda,$$

modulo the subspace generated by the elements

$$[T] - \operatorname{sgn}(q)[T],\ q \in C(T)$$

This is again a $\mathbb{C}S_n$-module by extending the action on numberings. Define $\widetilde{S}_\lambda$ as the subspace generated by

$$\widetilde{v}_T = a_T \cdot [T], T \text{ a numbering of } \lambda$$

This is again an irrep of S_n, and all irreps arise this way. A basis is again given by the $\widetilde{v}_T$, for T a standard tableau. There are canonical surjections of $\mathbb{C}S_n$-modules

$$\alpha : \widetilde{M}^\lambda \to S^\lambda : [T] \mapsto v_T,$$

and

$$\beta : M^\lambda \to \widetilde{S}^\lambda : \{T\} \mapsto \widetilde{v}_T$$

such that $\alpha|_{\widetilde{S}^\lambda}$ and $\beta|_{S^\lambda}$ are isomorphisms. This shows that

$$S^\lambda = \operatorname{Im}(\widetilde{M}^\lambda \to M^\lambda).$$

Representations of $GL(E)$

For E a complex vector space of dimension m, the n-th tensor power carries a right S_n-action

$$(u_1 \otimes \cdots \otimes u_n) \cdot \sigma = u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)}$$

For any (left) representation M of S_n, define

$$E(M) = E^{\otimes n} \otimes_{\mathbb{C}S_n} M = E^{\otimes n} \otimes_{\mathbb{C}} M \big/ \langle (w \cdot \sigma) \otimes v - w \otimes (\sigma \cdot v) \rangle$$

Now $E^{\otimes n}$ also carries a left $GL(E)$-action

$$g \cdot (u_1 \otimes \cdots \otimes u_n) = g \cdot u_1 \otimes \cdots \otimes g \cdot u_n,$$

which commutes with the S_n-action, so $GL(E)$ also acts from the left on $E(M)$

$$g \cdot (w \otimes v) = (g \cdot w) \otimes v$$

All of these representations are polynomial representations, and this construction is functorial. Some examples:

$$\begin{aligned} M = \text{trivial representation} &\mapsto E(M) = \mathrm{Sym}^n(E) \\ M = \text{sign representation} &\mapsto E(M) = \wedge^n(E) \\ M = \text{regular representation} &\mapsto E(M) = E^{\otimes n} \end{aligned}$$

Using these examples, and the property $E(N \circ M) = E(N) \otimes E(M)$, where

$$N \circ M = \mathrm{Ind}_{S_n \times S_m}^{S_{n+m}}(N \otimes M) = \mathbb{C}S_{n+m} \otimes_{\mathbb{C}[S_n \times S_m]} (N \otimes_{\mathbb{C}} M),$$

it follows that

$$\begin{aligned} E(M^\lambda) &= \mathrm{Sym}^{\lambda_1}(E) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_k}(E) \\ E(\widetilde{M}^\lambda) &= \wedge^{\mu_1}(E) \otimes \cdots \otimes \wedge^{\mu_l}(E) \end{aligned}$$

where $\mu = (\mu_1, \ldots, \mu_l)$ denotes the conjugate of $\lambda = (\lambda_1, \ldots, \lambda_k)$. So we have that

$$E(S^\lambda) = \mathrm{Im}(\wedge^{\mu_1}(E) \otimes \cdots \otimes \wedge^{\mu_l}(E) \to E^{\otimes n} \to \mathrm{Sym}^{\lambda_1}(E) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_k}(E))$$

Remember from the representation theory of Lie groups that a finite dimensonal, holomorphic representations of $GL_m(\mathbb{C})$ is irreducible if and only if it has a unique highest weight vector, up to scalar multiplication. Moreover, two representations are isomorphic if and only if their highest weight vectors have the same weight. For $E(S^\lambda)$, one can show that the unique highest weight vector is $e_1^{\otimes \lambda_1} \otimes \cdots \otimes e_m^{\otimes \lambda_m} \otimes v_T$, where T is any numbering of λ. One then deduces the following theorem

Theorem 5.2.2 *Consider $E = \mathbb{C}^m$.*

- *If λ has at most m rows, then the representation $E(S^\lambda)$ of $GL_m(\mathbb{C})$ is an irreducible representation with highest weight $\lambda = (\lambda_1, \ldots, \lambda_m)$. These are all of the irreducible polynomial representations of $GL_m(\mathbb{C})$.*

- *For any $\alpha = (\alpha_1, \ldots, \alpha_m)$, where $\alpha_1 \geq \ldots \geq \alpha_m$ are integers, there is a unique irreducible representation of $GL_m(\mathbb{C})$ with highest weight α, which can be realized as $E(S^\lambda) \otimes \det^{\otimes k}$, for any $k \in \mathbb{Z}$ with $\lambda_i = \alpha_i - k \geq 0$ for all i. In this realization, det denotes the determinant representation.*

- *All finite dimensional holomorphic representation of $GL_m(\mathbb{C})$ are rational.*

Proof: See Fulton and Harris [34], Lecture 15, §15.5. □

The endofunctors, indexed by partitions λ and given by

$$\mathbb{S}_\lambda : \mathrm{Vect} \to \mathrm{Vect} : E \mapsto E(S^\lambda)$$

are called Schur functors. Note that if λ has more rows than the dimension of a $V \in \mathrm{Vect}$, then the image will be zero.

Schur-Weyl duality

The connection between representation of S_n and representations of $GL(E)$ is known as Schur-Weyl duality. Though we have just seen a way of obtaining all polynomial representations of $GL(E)$ from those of S_n, later on we will be interested in more abstract versions of Schur-Weyl duality. It is for this reason that we now take a different approach, using the double commutant theorem and in particular the semisimplicity of $\mathbb{C}S_n$. Also, in this subsection we will give complete proofs. The classical reference for this material is Weyl [81], but we will use Etingof's notes on representation theory [28] and a very nice exposition due to Qiaochu Yuan [82].

Given an abelian group A and a collection of endomorphisms $T = \{T_i : A \to A\}$, the commutant T' of T is based on the centralizer in group theory:

$$T' = \{S \in \mathrm{End}(A) \mid T_i S = S T_i, \text{ for all } T_i \in T\}.$$

Since T' is also the commutant of the subring of $\mathrm{End}(A)$ generated by all T_i, we can assume T is such a subring, and then T' is just the ring of all endomorphisms of A as left T-module. As an example of what we want to do, consider a finite group G and a finite dimensional complex representation E, then according to Maschke, we have a decomposition

$$E = \bigoplus_i n_i E_i,$$

which is not canonical if $n_i > 1$. If, for example, G acts trivially on V, then V is a direct sum of copies of the trivial representation, and actually choosing a decomposition is equivalent to choosing a basis of V. There is however a way of specifying a representation in terms of its irreducible subrepresentations that is canonical, not depending on any choices. The multiplicities n_i can be described more intrinsically as $\dim(\mathrm{Hom}_G(E_i, E))$. Taking this as motivation, we replace n_i by the vector space $\mathrm{Hom}_G(E_i, E)$. These come with evaluation maps

$$E_i \otimes \mathrm{Hom}_G(E_i, E) \to E,$$

the images of which are exactly the E_i-isotypic components of E, providing an alternate and canonical decomposition

$$E = \bigoplus_i E_i \otimes \mathrm{Hom}_G(E_i, E).$$

The double commutant theorem discusses the structure present on these 'multiplicity spaces' $\mathrm{Hom}_G(E_i, E)$. They are not only vector spaces, but also have the structure of $\mathrm{End}_G(E)$-modules. Lastly, $\mathrm{End}_G(E)$ is exactly the commutant of the image of $\mathbb{C}G$ in $\mathrm{End}(E)$.

Theorem 5.2.3 (Double commutant theorem) *For an abelian group A and a subring $T \subset \mathrm{End}(A)$ such that*

1. *T is a semisimple ring*
2. *A is a finite direct sum of simple T-modules*

one has that $T = T''$. Moreover, T' is semisimple, and as $T \otimes T'$-modules there is a decomposition

$$A = \bigoplus_i M_i \otimes_{D_i} N_i,$$

where the M_i (respectively N_i) are all simple T-modules (respectively, simple T'-modules), and the $D_i = \mathrm{End}_T(M_i) = \mathrm{End}_T(N_i)^{op}$ are division rings.

Proof: Choose a finite direct sum decomposition

$$A = \bigoplus_i n_i M_i,$$

where the M_i are the simple T-modules. The action of T on A is faithful, and knowing the structure of T from the Artin-Wedderburn theorem, we deduce that all n_i are strictly positive. Schur's lemma provides the isomorphism

$$\mathrm{End}_T(A) = T' \cong \prod_i M(n_i, D_i),$$

where the $D_i = \mathrm{End}_T(M_i)$ are division rings, showing that T' is semisimple. Using the action of T' on the multiplicity spaces $\mathrm{Hom}_T(M_i, A)$, and the two decompositions above, these are exactly the simple T'-modules. These $N_i = \mathrm{Hom}_T(M_i, A)$ are in fact the unique, simple T'-modules on which $M(n_i, D_i)$ acts nontrivially. This means N_i is an n_i-dimensional D_i^{op}-vector space, and just as in the case of finite groups, we have an isomorphism

$$\bigoplus_i M_i \otimes_{D_i} N_i \xrightarrow{\cong} A.$$

Turning this around, we can think of M_i as the multiplicity spaces of the decomposition of A as a T'-modules, so A is also the finite direct sum of simple T'-modules, and using Artin-Wedderburn again, we conclude that $T'' = T$. $\square$

Corollary 5.2.4 *There is a canonical bijection between simple T-modules and simple T'-modules.*

Now we'll use this theorem to prove Schur-Weyl duality. Given a finite dimensional complex vector space E, we already noted that the symmetric group S_n acts on $E^{\otimes n}$ by permuting the factors, and that this action commutes with the diagonal action of $GL(E)$, giving a morphism

$$\mathbb{C}S_n \to \mathrm{End}_{GL(E)}(E^{\otimes n}),$$

and we can ask whether this morphism is surjective. Schur-Weyl duality answers this question in the positive. We will first prove the theorem for $\mathfrak{gl}(E)$, the Lie algebra of $GL(E)$, where E can be any finite dimensional vector space over a field k of characteristic 0. For V, W representations of $\mathfrak{gl}(E)$, the tensor product $V \otimes W$ is a also a representation by

$$X(v \otimes w) = Xv \otimes w + v \otimes Xw,$$

coming from the usual action of e^{tX} on a tensor product.

Lemma 5.2.5 *If, for all $t \in k$, E contains*

$$f(t) = \sum_{i=0}^{d} e_i t^i,$$

where e_i are vectors in a vector space containing E, then E contains $e_0, \ldots, e_n$.

Proof: For $d = 0$, this is obvious. Setting $t = 0$, we get $e_0 \in E$. For $t \neq 0$, one has

$$\frac{f(t) - f(0)}{t} = \sum_{i=1}^{d} e_i t^{i-1} \in E.$$

By Lagrange interpolation, and since we work over an infinite field, any value of a polynomial of degree d is a linear combination of values that the polynomial attains at any $d+1$ distinct points. We conclude that E contains $\frac{f(t)-f(0)}{t}$ for all t, which is a polynomial of degree $d-1$, and we can use induction. □

Lemma 5.2.6 *The symmetric power $S^n E$ is spanned by elements of the form e^n for $e \in E$.*

Proof: Take F to be the subspace spanned by the e^n, and let $e_1, \ldots, e_n$ be a basis of E. By definition, F contains

$$\left(\sum_i t_i e_i\right)^n = \sum_{\sum_i m_i = n} \binom{n}{m_1, \ldots, m_n} \prod_i t_i^{m_i} e_i^{m_i},$$

for all $t_i \in k$. Using the previous Lemma n times, F contains $\prod_i e_i^{m_i}$ for all m_i such that $\sum_i m_i = n$, and these form a basis for $S^n E$. □

Lemma 5.2.7 *Suppose A is a finite dimensional algebra over a field k of characteristic 0. The invariant subalgebra $(A^{\otimes n})^{S_n}$ is generated as an algebra by all elements of the form*

$$\Delta_n(a) = a \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes a \otimes \cdots \otimes 1 + \ldots + 1 \otimes 1 \otimes \cdots \otimes a.$$

Proof: We use a classic theorem that says that the ring of symmetric polynomials with coefficients in a field of characteristic 0 is a commutative polynomial ring in the power sum symmetric polynomials $p_k(x_1, \ldots, x_n) = x_1^k + \ldots + x_n^k$, k ranging from 1 to n; see Fulton and Harris [34], Appendix A. In particular, the elementary symmetric polynomial $e_n = x_1 x_2 \cdots x_n$ can be written this way, whence we conclude that $a \otimes a \otimes \cdots \otimes a$ is a polynomial in the elements $\Delta_n(a), \Delta_n(a^2), \ldots, \Delta_n(a^n)$. Again, in charactertistic 0, $(A^{\otimes n})^{S_n}$ can be identified with $S^n A$, and the previous Lemma provides the conclusion. □

Theorem 5.2.8 (Schur-Weyl duality for $\mathfrak{gl}(E)$) *The natural map $kS_n \to \mathrm{End}_{\mathfrak{gl}(E)}(E^{\otimes n})$ is surjective.*

Proof: Elements of $\mathfrak{gl}(E)$ act by the Leibniz rule on $E^{\otimes n}$:

$$X(e_1 \otimes \cdots \otimes e_n) = \sum_i e_1 \otimes \cdots \otimes Xe_i \otimes \cdots \otimes e_n.$$

Denote the image under this action of $\mathfrak{U}(\mathfrak{gl}(E))$, the universal enveloping algebra of $\mathfrak{gl}(E)$, in $\mathrm{End}(E^{\otimes n})$ by U. Then elements of $\mathrm{End}_{\mathfrak{gl}(E)}(E^{\otimes n})$ correspond to U'. For $T = \mathrm{Im}(kS_n)$, and $A = E^{\otimes n}$, we see that the conditions of the double commutant theorem are satisfied, since kS_n is semisimple, so $\mathrm{Im}(kS_n) = (\mathrm{Im}(kS_n))''$. To prove the theorem, it thus suffices to show that $(\mathrm{Im}(kS_n))' = U$. It is clear that $(\mathrm{Im}(kS_n))' = \mathrm{End}(E^{\otimes n})^{S_n} \cong (\mathrm{End}(E)^{\otimes n})^{S_n}$, with respect to the obvious action of S_n. Using the previous lemma it immediately follows that this invariant subalgebra is generated by elements

$$e_1 \otimes \cdots \otimes e_n \mapsto \sum_i e_1 \otimes \cdots \otimes Xe_i \otimes \cdots \otimes e_n,$$

and the theorem is proved. □

Theorem 5.2.9 *The subalgebra of $\mathrm{End}(E^{\otimes n})$ spanned by elements of $\mathfrak{gl}(E)$ is exactly the subalgebra of $\mathrm{End}(E^{\otimes n})$ spanned by elements of $GL(E)$.*

Proof: We already know that the subalgebra spanned by $GL(E)$ is contained in $(\mathrm{Im}(kS_n))'$, and the previous version of Schur-Weyl duality tells us that it is contained in the subalgebra spanned by $\mathfrak{gl}(E)$. The other inclusion uses Lagrange interpolation again: for an $X \in \mathfrak{gl}(E)$, the element $(t + X)^{\otimes n}$ is contained in the subalgebra spanned by $GL(E)$ for almost all t, implying this also holds for $t = 0$. Lemma 5.2.6 then concludes the proof. □

Corollary 5.2.10 (Schur-Weyl duality for $GL(E)$) *The natural map $kS_n \to \mathrm{End}_{GL(E)}(E^{\otimes n})$ is surjective.*

The second part of the double commutant theorem implies there is a decomposition

$$E^{\otimes n} = \bigoplus_{\substack{\lambda \text{ partition} \\ \text{of } n}} S^\lambda \otimes \mathrm{Hom}_{S_n}(S^\lambda, V^{\otimes n}),$$

such that each $E^\lambda := \mathrm{Hom}_{S_n}(S^\lambda, E^{\otimes n})$ is an irrep of $GL(E)$, or 0. This provides another way of defining the Schur functors

$$\mathbb{S}_\lambda : \mathrm{Vect} \to \mathrm{Vect} : E \mapsto E^\lambda,$$

which is in fact identical to the one defined before.

Symmetric functions

The fundamental theorem of symmetric functions (see Fulton and Harris [34], Appendix A) says that $\mathbf{Symm}_m = \mathbb{Z}[x_1, \ldots, x_m]^{S_m}$ is a polynomial algebra in the elementary symmetric polynomials $\Lambda_k, k = 1, \ldots, m$, where

$$\Lambda_k(x_1, \ldots, x_m) = \sum_{1 \leq j_1 < \cdots < j_k \leq m} x_{j_1} \cdots x_{j_k}.$$

Other families of generators are given by the complete homogeneous symmetric polynomials

$$S_k(x_1, \ldots, x_m) = \sum_{j_1 + \ldots + j_m = k; j_i \geq 0} x_1^{j_1} \cdots x_m^{j_m},$$

or the power sum symmetric polynomials

$$\Psi_k(x_1, \ldots, x_m) = \sum_{j=1}^{k} x_i^k.$$

Bases are obviously formed by monomials in these generators, but also by another type of symmetric functions, indexed by partitions $\lambda = (\lambda_1, \ldots, \lambda_m), \lambda_1 \geq \cdots \geq \lambda_m > 0$: the Schur functions, defined by

$$S_\lambda = \sum_T \prod_{i=1}^{m} x_i^{\#\text{ of times } i \text{ occurs in } T},$$

where the sum is taken over all Young tableaux on λ, using numbers $1, \ldots, m$. This turns S_λ into a symmetric homogeneous polynomial of degree $|\lambda| = \sum_i \lambda_i$. We already know that the polynomial representations of $GL_m = GL_m(\mathbb{C})$ are indexed by partitions λ with $\leq m$ rows. The character of such a representation V in a matrix $g \in GL_m$, denoted $\chi_V(g)$, can be considered as a polynomial function in the eigenvalues x_i of g, since this is true for diagonal matrices, which are dense in GL_m. Moreover, $\chi_V(g)$ is symmetric, because the character is a class function. In fact, one has the following theorem:

Theorem 5.2.11 *The irreducible polynomial characters of GL_m are exactly the S_λ, for λ a partition with at most m rows.*

Proof: See Fulton [35], Chapter 8, §8.3. □

Denote by R_n the free abelian group on isomorphism classes of irreps of S_n, or equivalently, the Grothendieck group of representations of S_n, i.e. the free abelian group on the isomorphism classes $[V]$ of representations with relations given by $[V] + [W] = [V \oplus W]$. Set $R = \oplus_{n=0}^{\infty} R_n$, where $R_0 = \mathbb{Z}$. Then R can be given a (commutative, graded, unital) ring structure via

$$\circ : R_n \times R_m \to R_{n+m} : ([V], [W]) \mapsto [V] \circ [W] = [\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} V \otimes W],$$

and $S_n \times S_m$ is considered a subgroup of S_{n+m} in canonical fashion. Coming back to the symmetric polynomials, one can easily check that all the generators we defined are independent of how many variables are used. More precisely, for $l < m$ one has

$$p(x_1, \ldots, x_l, 0, \ldots, 0) = p(x_1, \ldots, x_l),$$

for any of these polynomials, if we consider them with the proper number of variables. A symmetric function of degree n is then defined to be a collection of symmetric polynomials $p(x_1, \ldots, x_m)$ of degree n, that satisfy the displayed identity. One does this for the simple reason that it is sometimes convenient to have sufficiently many variables; the Schur polynomials for example, are not defined (or vanish) if the number of variables is smaller than the number of parts of the partition. The ring of symmetric functions is then defined by $\mathbf{Symm} = \oplus_{m=0}^{\infty} \mathbf{Symm}_m \cong \mathbb{Z}[x_1, \ldots, x_m, \ldots]$. These definitions give rise to the following fundamental theorem

Theorem 5.2.12 *Define a linear map ϕ :$\mathbf{Symm} \to R$ by $\phi(S_\lambda) = [S^\lambda]$. This map is then an isomorphism of graded rings.*

Proof: See Fulton [35], Chapter 7, §7.3. □

This theorem allows one to deduce properties of representations of symmetric groups from known properties of symmetric functions. The representation ring of GL_m, denoted $\mathcal{R}(m)$, is defined to be the Grothendieck ring of polynomial (or rational) representations. Because of complete reducibility, this is also the free abelian group on isomorphism classes of irreps. This time the product structure comes from the tensor product of representations: $[V] \cdot [W] = [V \otimes_{\mathbb{C}} W]$. Using the $E(-)$ construction, one gets an additive homomorphism $R_n \to \mathcal{R}(m)$, and by adding over n, a morphism from R to $\mathcal{R}(m)$. This can be composed with the character map to land in the ring $\mathbf{Symm}_m$ of symmetric polynomials in the variables $x_1, \ldots, x_m$. Summarizing, we get a sequence of maps

$$\mathrm{Symm} \to R \to \mathcal{R}(m) \to \mathrm{Symm}_m : S_\lambda \mapsto [S^\lambda] \mapsto [E(S^\lambda)] \mapsto S_\lambda(x_1, \ldots, x_m),$$

with $|\lambda| = n$. Using the fact that the Schur functions form a basis for $\mathbf{Symm}$, $\mathcal{R}(m) \to \mathbf{Symm}_m$ is an isomorphism of rings, showing that the representation ring of GL_m is actually isomorphic to $\mathbb{Z}[\Lambda_1, \ldots, \Lambda_m]$ (for rational representations, one also gets the inverse of Λ_m, corresponding to the determinant, so that we have $\mathbb{Z}[\Lambda_1, \ldots, \Lambda_m]_{\Lambda_m}$).

5.3 The bialgebra $\mathcal{O}_{\mathrm{nc}}(M_n)$

We will apply the material of the previous chapters to the symmetric algebra $A = SV = TV/(\wedge^2 V)$ of a vector space V of dimension n. In this case, $N = 2$, and equipping V with the basis $(x_i)_i$, and putting $z_i^j = x_j^* \otimes x_i$, the coordinate ring of the algebraic monoid of $n \times n$-matrices is given by $\mathcal{O}(M_n) = k[(z_i^j)_{i,j}] = S(\mathrm{End}(V)^*)$. This is a bialgebra and SV is a comodule, with formulas given by

$$\begin{aligned} \Delta(z_i^j) &= \sum_k z_i^k \otimes z_k^j \\ \epsilon(z_i^j) &= \delta_i^j \\ \delta(x_i) &= \sum_k z_i^k \otimes x_k, \end{aligned}$$

just like in Section 4.1. Applying the construction of that section to A, we see that

$$\underline{\mathrm{end}}(A) = k\langle z_1^1, \ldots, z_n^n \rangle \Big/ \left(\begin{matrix} z_i^k z_j^k = z_j^k z_i^k, \quad [z_i^j, z_k^l] = [z_k^j, z_i^l] \\ i, j, k, l \in \{1, \ldots, n\} \end{matrix} \right)$$

and the same formulas for Δ, ϵ and δ. Since there is a surjective bialgebra morphism

$$B : \underline{\mathrm{end}}(A) \to \mathcal{O}(M_n) : z_i^j \mapsto z_i^j$$

that is compatible with the coactions on SV, we denote $\underline{\mathrm{end}}(A)$ by $\mathcal{O}_{\mathrm{nc}}(M_n)$, and think of it as a non-commutative version of the coordinate ring of $n \times n$-matrices. The pushforward on comodules provides an exact monoidal functor

$$B_* : \mathrm{CoMod}(\mathcal{O}_{\mathrm{nc}}(M_n)) \to \mathrm{CoMod}(\mathcal{O}(M_n)).$$

In the following, we would like to make a link with Young diagrams, as they appeared in previous sections. The description of the Schur functors $\mathbb{S}_\lambda$ given in previous sections allows for a generalization to skew Young diagrams λ/μ, where λ and μ are both ordinary Young diagrams and $\lambda_i \geq \mu_i$. For example, using $\lambda = (3,3,1)$ and $\mu = (2,1)$, the corresponding picture is given by

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 \\ \cline{1-1}\end{array} \; - \; \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1}\end{array} \; = \; \begin{array}{cccc} & & & \boxed{3} \\ & & \boxed{5} & \boxed{6} \\ \boxed{7} & & & \end{array}$$

Skew Young symmetrizers can be defined in $kS_d, d = \sum_i \lambda_i - \mu_i$, exactly the same way as before, providing us with a representation of S_d. Skew Schur functors are analogously defined

$$\mathbb{S}_{\lambda/\mu} V = \mathrm{Im}(\wedge^{\lambda'_1 - \mu'_1} V \otimes \cdots \otimes \wedge^{\lambda'_l - \mu'_l} V \xrightarrow{\alpha} V^{\otimes d} \xrightarrow{\beta} S^{\lambda_1 - \mu_1} V \otimes \cdots \otimes S^{\lambda_l - \mu_l} V),$$

where λ' and μ' are the respective conjugate partitions.

Remark 5.3.1 *These representations are not irreducible in general, and the decomposition is given by*

$$\mathbb{S}_{\lambda/\mu} V = \sum N_{\mu\nu\lambda} \mathbb{S}_\nu V,$$

where the $N_{\mu\nu\lambda}$ are the Littlewood-Richardson coefficients, that can be determined in a combinatorial fashion from the occurring Young tableaux, see Fulton [34], Chapter $6, \S 6.1$.

This allows for a formulation of the theorem relating $\mathcal{O}_{nc}(M_n)$ to skew Schur functors.

Proposition 5.3.2 *For $A = k[x_1, \ldots, x_n]$, all irreducibles of $\underline{\mathrm{end}}(A)$ correspond to images of a skew Schur functor under the pushforward of the map $B : \underline{\mathrm{end}}(A) \to \mathcal{O}(M_n)$.*

Proof: The basic cases are obvious:

$$B_*(\mathcal{F}_A(S^l_{\{1,\ldots,l\}})) = B_*(\wedge^l V) = \wedge^l V$$
$$B_*(\mathcal{F}_A(S^l_\emptyset)) = B_*(S^l V) = S^l V$$

For general $I \subset \{1, \ldots, m\}$, switch to a presentation, apply $\mathcal{F}_A$ and use distributivity. This gives

$$\mathcal{F}_A(S^m_I) = R^{m+1}_I \big/ \big(R^{m+1}_I \cap \sum_{i \notin I} R^{m+1}_i\big),$$

or in a notation more in tune with the skew Schur functors

$$\mathcal{F}_A(S^m_I) = \mathrm{Im}\left(R^{m+1}_I \to V^{\otimes m+1} \to V^{\otimes m+1} \Big/ \sum_{i \notin I} R^{m+1}_i\right)$$

Now since we're working with $A = SV$, R is spanned by $x_i x_j - x_j x_i$ where the $(x_i)_i$ form a basis of V. To continue, we need rim hooks: these are special types of skew Young diagrams that do not contain any 2×2-boxes. We denote rim hooks by the number of elements in each row, just like other Young diagrams. For example, the rim hook corresponding to $J = (1, 1, 3, 1)$ is

$$\begin{array}{|c|c|c|}
\cline{3-3}
\multicolumn{2}{c|}{} & 1 \\
\cline{3-3}
\multicolumn{2}{c|}{} & 2 \\
\hline
3 & 4 & 5 \\
\hline
6 & \multicolumn{2}{c}{} \\
\cline{1-1}
\end{array}$$

Now for any subset $I \subset \{1, \ldots, m\}$, rewrite I in the form

$$I = \{i_1, i_1 + i_2, i_1 + i_2 + i_3, \ldots, \sum_{j=1}^{k} i_j\},$$

and consider the rim hook with $m + 1$ boxes given by $\widetilde{I} = (i_1, i_2, i_3, \ldots, i_k, m + 1 - i_k)$. For example, the rim hook corresponding to $I = \{2, 5\} = \{2, 2 + 3\} \subset \{1, \ldots 5\}$ would be

$$\begin{array}{|c|c|c|c|}
\cline{3-4}
\multicolumn{2}{c|}{} & 1 & 2 \\
\hline
3 & 4 & 5 & \multicolumn{1}{c}{} \\
\cline{1-3}
6 & \multicolumn{3}{c}{} \\
\cline{1-1}
\end{array}$$

Now we have

$$\begin{aligned}
R_I^{m+1} &= \wedge^{i'_1} V \otimes \wedge^{i'_2} V \otimes \cdots \otimes \wedge^{i'_p} V \\
V^{\otimes m+1} / \sum_{i \notin I} R_i^{m+1} &= S^{i_1} V \otimes S^{i_2} V \otimes \cdots \otimes S^{m+1-i_k} V
\end{aligned}$$

where $i' = (i'_1, \ldots, i'_p)$ denotes the conjugate partition of $m+1$. This corresponds precisely to the skew Schur functor $S^{\lambda/\mu} V$, where λ/μ is the rim hook we defined. $\square$

6

Some ideas for further study

> "If it disagrees with experiment, it's WRONG. That's all there is to it."
>
> *Lecture at Cornell University, 1964*
> Richard Phillips Feynman

This chapter is a mixture of thoughts and minor results obtained by Michel Van den Bergh and the author in the fall and winter of 2012. I am greatly indebted to him for his generous sharing of ideas and proofs, and for answering many of my questions. Only the material that is more or less presentable has been included. I feel very strongly, perhaps too *stringently*, that one can judge the importance of a part of mathematics by the amount of interaction it has with other parts of the subject. A possible connection between the $\mathcal{O}_{nc}$-construction and the Markoff conjecture in number theory was suggested by Van den Bergh, though we do not speak of it here. This most exciting possibility will have to be taken up in the future, as will the connections with the theory of quasi-hereditary (co)algebras.

6.1 Non-commutative symmetric functions

The last chapter motivates the following 'experiment': if $\mathcal{O}_{nc}(M_n)$ were to be a good candidate for a non-commutative symmetry group, it should support nice generalizations of the commutative theory. In particular, the representation ring $\mathcal{R}(\mathcal{O}_{nc}(M_n))$ should correspond to the obvious non-commutative generalization of $\mathbb{Z}[\Lambda_1, \ldots, \Lambda_n]$, namely $\mathbb{Z}\langle\Lambda_1, \ldots, \Lambda_n\rangle$, the free polynomial ring in n variables. We intend to prove this statement. In Gelfand et al. [37] and consecutive papers, a theory of non-commutative symmetric functions is built up, starting from the definition of the ring of these functions as $\mathbf{Symm_{nc}} = \mathbb{Z}\langle\Lambda_1, \ldots, \Lambda_n, \ldots\rangle$, the Λ_i being formal variables, interpreted as non-commutative elementary symmetric functions. For us, it is important that there are analogues of the commutative Schur functions: in the classic setting, the Schur functions indexed by partitions form a $\mathbb{Z}$-basis of **Symm**. In the non-commutative case, a similar result holds true if one considers so called ribbon Schur functions.

Definition 6.1.1 *Given a rim hook (or ribbon diagram) $I \in (\mathbb{N}^*)^k$, with conjugate diagram $I' = (j_1, \ldots, j_l)$. Then the ribbon Schur function R_I is defined by the quasi-determinant [37]*

$$R_I = (-1)^{l-1} \begin{vmatrix} \Lambda_{j_l} & \Lambda_{j_{l-1}+j_l} & \Lambda_{j_{l-2}+j_{l-1}+j_l} & \cdots & \underline{\Lambda_{j_1+\ldots+j_l}} \\ 1 & \Lambda_{j_{l-1}} & \Lambda_{j_{l-2}+j_{l-1}} & \cdots & \Lambda_{j_1+\ldots+j_{l-1}} \\ 0 & 1 & \Lambda_{j_{l-2}} & \cdots & \Lambda_{j_1+\ldots+j_{l-2}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda_{j_1} \end{vmatrix}$$

If the underlying matrix is denoted A, then $R_I = |A|_{1l}$, which is calculated by

$$|A|_{1l} = A_{1l} - (A_{11}, A_{12}, \ldots, \widehat{A_{1l}}) \cdot (A^{1l})^{-1} \cdot (\widehat{A_{1l}}, A_{2l}, \ldots, A_{ll})^t,$$

upper indices denoting cofactors.

This allows us to connect non-commutative symmetric functions to the representation theory of $\mathcal{O}_{nc}(M_n)$. As usual, by $\mathcal{R}(\mathcal{O}_{nc}(M_n))$ we denote the representation ring of $\mathcal{O}_{nc}(M_n)$: it is spanned as an abelian group by isomorphism classes of simple representations (comodules) with sum induced by the direct sum $[M] + [N] = [M \oplus N]$, and product induced by the tensor product of representations $[M] \cdot [N] = [M \otimes N]$.

Theorem 6.1.2 *There is a surjective ring morphism $\Phi : \mathbf{Symm_{nc}} \to \mathcal{R}(\mathcal{O}_{nc}(M_n))$ such that $\mathrm{Ker}(\Phi) = (\Lambda_{n+1}, \ldots, \Lambda_{n+k}, \ldots)$. In other words,*

$$\mathcal{R}(\mathcal{O}_{nc}(M_n)) \cong \mathbb{Z}\langle \Lambda_1, \ldots, \Lambda_n \rangle$$

Proof: According to Section 4 of [37], the ring of non-commutative symmetric functions has the ribbon Schur functions R_I as a linear basis, where $I = (i_1, \ldots, i_k)$ has the row lengths of the ribbon diagram as its components (thus the diagram has $m = \sum_j i_j$ blocks). For example, $R_{(2,2)}$ corresponds to the diagram

$$\begin{array}{|c|c|c|} \hline & 1 & 2 \\ \hline 3 & 4 & \\ \cline{1-2} \end{array}$$

Defining Φ in the following way

$$\begin{aligned} \Phi : \mathbf{Symm_{nc}} &\to \mathcal{R}(\mathcal{O}_{nc}(M_n)) \\ R_I &\mapsto C^m_{\widetilde{I}}, \end{aligned}$$

where $\widetilde{I} = \{i_1, i_1 + i_2, i_1 + i_2 + i_3, \ldots, i_1 + \ldots + i_{k-1}\}$, and the C^m_T are the simple $\mathcal{O}_{nc}(M_n)$-comodules, defined as in Corollary 4.3.11 , we get a linear map. To check that it is a well defined ring morphism, we use Proposition 3.13 of [37], which contains the basis expansion of the product of ribbon Schur functions. For $I = (i_1, \ldots, i_k)$ such that $\sum_t i_t = p$, and $J = (j_1, \ldots, j_l)$ such that $\sum_t j_t = q$, it says that

$$R_I \cdot R_J = R_{I \rhd J} + R_{I \cdot J}, \tag{6.1}$$

where $I \rhd J = (i_1, \ldots, i_{k-1}, i_k + j_1, \ldots, j_l)$ and $I \cdot J = (i_1, \ldots, i_k, j_1, \ldots, j_l)$. Pictorially, this corresponds to horizontal (respectively vertical) concatenation. For example, for $I = (1), J = (2)$, we get (suppressing the R's)

$$\boxed{1} \cdot \boxed{2}\boxed{3} = \boxed{1}\boxed{2}\boxed{3} + \begin{array}{|c|c|c|} \cline{3-3} & & 1 \\ \hline 2 & 3 & \\ \cline{1-2} \end{array}$$

Knowing this formula, it suffices to prove that $\Phi(R_{I\rhd J}) + \Phi(R_{I\cdot J}) = \Phi(R_I)\cdot\Phi(R_J)$. The right hand side is equal to $C^n_{\widetilde{I}} \otimes C^m_{\widetilde{J}}$, which corresponds via Corollary 4.3.12 to the tensor product of simple quiver representations $S^{n-1}_{\widetilde{I}} \otimes S^{m-1}_{\widetilde{J}}$. Using Proposition 3.3.3 for the case $N = 2$, we see that the Jordan-Hölder filtration of this product is made up out of $S^{m+n-1}_{K_1}$ and $S^{m+n-1}_{K_2}$, for

$$K_1 = \{i_1, i_1+i_2, \ldots, i_1+\ldots+i_{k-1}, n+j_1, n+j_1+j_2, \ldots, n+j_1+\ldots+j_l\}, \text{ and}$$

$$K_2 = \{i_1, i_1+i_2, \ldots, i_1+\ldots+i_{k-1}, n, n+j_1, n+j_1+j_2, \ldots, n+j_1+\ldots+j_l\} = K_1 \cup \{n\}.$$

As for the left hand side, we see that

$$\Phi(R_{I\rhd J}) = C^{n+m}_{\widetilde{I\rhd J}}, \quad \widetilde{I\rhd J} = \{i_1, i_1+i_2, \ldots, i_1+\ldots+i_{k-1}, i_1+\ldots+i_k+j_1, \ldots, i_1+\ldots+j_{l-1}\},$$

and

$$\Phi(R_{I\cdot J}) = C^{n+m}_{\widetilde{I\cdot J}}, \quad \widetilde{I\cdot J} = \{i_1, i_1+i_2, \ldots, i_1+\ldots+i_{k-1}, i_1+\ldots+i_k, i_1+\ldots+i_k+j_1, \ldots, i_1+\ldots+j_{l-1}\}.$$

These correspond as quiver representations to $S^{m+n-1}_{\widetilde{I\rhd J}}$ and $S^{m+n-1}_{\widetilde{I\cdot J}}$, and $\widetilde{I\rhd J} = K_1$, $\widetilde{I\cdot J} = K_2$, proving Φ is a ring morphism.

The surjectivity of Φ follows immediately from Proposition 5.3.2, since all simples of $\mathcal{O}_{nc}(M_n)$ can be realized as rim hooks. We claim that the induced map

$$\Phi : \mathbf{Symm}_{nc}/(\Lambda_{n+1}, \ldots) \to \mathcal{R}(\mathcal{O}_{nc}(M_n))$$

is an isomorphism. To show that it is well defined, first notice that $\Lambda_k = R_{(1,1,\ldots,1)}$, where 1 appears k times. Under Φ, this gets mapped to $C^{k+1}_{\{1,\ldots,k\}}$. Using the definition of the C's, see Definition 4.3.8, this is

$$C^{k+1}_{\{1,\ldots k\}} = R^{k+1}_{\{1,\ldots,k\}} = (R \otimes V^{\otimes k-1}) \cap (V \otimes R \otimes V^{\otimes k-2}) \cap \cdots \cap (V^{\otimes k-1} \otimes R).$$

For a commutative polynomial ring, R is the vector space spanned by the elements $x_ix_j - x_jx_i$, so elements in this intersection correspond to elements of $\wedge^{k+1}V \hookrightarrow V^{k+1}$. Since V is n dimensional, it immediately follows that $(\Lambda_{n+1}, \ldots, \Lambda_{n+k}, \ldots) \subset \mathrm{Ker}(\Phi)$. To show injectivity, we first show that $\mathbf{Symm}_{nc}/(\Lambda_{n+1}, \ldots)$ has a basis given by the images of the R_I such that the columns of I have length $\leq n$. Suppose the column lengths of a general I are denoted $j_1, \ldots, j_k$. Using formula (6.1), any R_I can be written as the product of its columns Λ_{j_l}, plus a linear combination of products of $R_{(1,1,\ldots,1)} = \Lambda_i$ of less than k factors. In each term of this linear combination, one of the factors will have length $\geq \max\{j_l\}$. Let's give an example: look at R_I, with I corresponding to

$$\begin{array}{ccc} & & \boxed{1} \\ & & \boxed{2} \\ & \boxed{3} & \boxed{4} \\ \boxed{5} & \boxed{6} & \end{array} = \boxed{5}\cdot\begin{array}{c}\boxed{3}\\ \boxed{6}\end{array}\cdot\begin{array}{c}\boxed{1}\\ \boxed{2}\\ \boxed{4}\end{array} - \left(\begin{array}{c}\boxed{3}\\ \boxed{6}\\ \boxed{5}\end{array}\cdot\begin{array}{c}\boxed{1}\\ \boxed{2}\\ \boxed{4}\end{array} + \boxed{5}\cdot\begin{array}{c}\boxed{1}\\ \boxed{2}\\ \boxed{4}\\ \boxed{3}\\ \boxed{6}\end{array} - \begin{array}{c}\boxed{1}\\ \boxed{2}\\ \boxed{4}\\ \boxed{3}\\ \boxed{6}\\ \boxed{5}\end{array}\right),$$

thus $R_I = \Lambda_1\Lambda_2\Lambda_3 - \Lambda_3^2 + \Lambda_1\Lambda_5 - \Lambda_6$. This shows that if any of the columns of I contains more than n elements, the corresponding R_I is zero in the quotient. Since an I is uniquely determined by its column lengths, this shows the claim. Notice that what we proved is essentially that the R_I have a triangular

expansion in terms of the standard basis $(\Lambda^I)_I$, providing an alternative way of seeing that they form a basis for $\mathbf{Symm}_{nc}$. Now $\mathcal{R}(\mathcal{O}_{nc}(M_n))$ has a basis given by the C_I, such that $C_I \neq 0$. To check that $C_I \neq 0$ if the corresponding skew diagram does not contain columns of length $> n$, it is sufficient to write down a non-zero element. This is directly seen to exist by the last few lines of the proof of Proposition 5.3.2, proving the theorem. □

6.2 Tannaka-Krein versus Schur-Weyl

Tannaka-Krein duality is the name given to a number of statements all related to reconstructing a specific type of algebraic structure from its category of representations, usually equipped with some additional data. This section provides a brief introduction, since the end construction from previous chapters can be viewed in this light, and suggests a natural weakening of Tannaka-Krein duality, which we will call Schur-Weyl duality. The first section is based on Etingof's notes [27], where one can also find the basic definitions and properties of monoidal and abelian categories. In this section, Vect will denote the category of finite dimensional vector spaces over a field.

Definition 6.2.1 *A k-linear abelian category $\mathcal{C}$ is finite if it is equivalent to the category of finite dimensional modules over some finite dimensional k-algebra A.*

A more intrinsic characterization of this concept also exists.

Proposition 6.2.2 *A k-linear abelian category $\mathcal{C}$ is finite iff the following conditions are satsified*

- *$\mathcal{C}$ has finite dimensional morphism spaces*
- *Every object has a finite composition series*
- *$\mathcal{C}$ has enough projectives*
- *There are finitely many isomorphism classes of simples*

If $\mathcal{C}$ is essentially small, and only satisfies the first two of these four properties, then it is said to be locally finite.

Proof: One direction is clear. The other is realized by taking $A = \mathrm{End}(P)^{op}$, where $P = \oplus_i P_i$, and the P_i are the projective covers of the simple objects. This represents a functor $F = \mathrm{Hom}(P, -) : \mathcal{C} \to \mathrm{Vect}$. Projectivity translates into exactness of F, and the generating property implies F is faithful. The algebra $A = \mathrm{End}(P)^{op}$ can alternatively be defined as $\mathrm{End}(F)$, the algebra of natural transformations $F \Rightarrow F$. Conversely, any exact faithful functor is represented by a unique projective generator up to unique isomorphism. See Etingof [27], Chapter 1, §1.18 for details. □

Definition 6.2.3 *A quasi-fiber functor on a k-linear abelian monoidal category is an exact faithful functor $F : \mathcal{C} \to \mathrm{Vect}$ that satisfies $F(1) = k$, and is equipped with an isomorphism $J : F(\bullet) \otimes F(\bullet) \to F(\bullet \otimes \bullet)$. If J is monoidal, then F is said to be a fiber functor.*

These definitions lead to the first reconstruction theorem. Given a fiber functor F, one can consider the algebra of natural transformations of F, $H = \mathrm{End}(F)$. Denoting by $\alpha_{F,F}$ the canonical algebra isomorphism

$$\alpha_{F,F} : \mathrm{End}(F) \otimes \mathrm{End}(F) \to \mathrm{End}(F \otimes F),$$

one checks that this algebra is in fact a bialgebra by

$$\Delta(a) = \alpha_{F,F}^{-1}(\widetilde{\Delta}(a)),$$

where

$$\widetilde{\Delta}(a)_{X,Y} = J_{X,Y}^{-1} \circ a_{X \otimes Y} \circ J_{X,Y}, \ X, Y \in \mathcal{C},$$

and counit $\epsilon(a) = a_1 \in k$. For any (not necessarily finite dimensional) bialgebra H, its category of finite dimensional representations $\mathrm{Rep}(H)$ is an abelian monoidal category, with the forgetful functor $F : \mathrm{Rep}(H) \to \mathrm{Vect}$ as fiber functor. These considerations lead to

Theorem 6.2.4 (Reconstruction theorem for finite dimensional bialgebras) *The assignments*

$$(\mathcal{C}, F) \mapsto H = \mathrm{End}(F), \ H \mapsto (\mathrm{Rep}(H), \mathrm{Forget})$$

are mutually inverse bijections between

- *Finite abelian k-linear monoidal categories $\mathcal{C}$ with a fiber functor F, up to monoidal equivalence and isomorphism of monoidal functors,*
- *Finite dimensional bialgebras H over k up to isomorphism.*

Proof: More or less clear from the above, modulo some computations; see Etingof [27], Chapter 1, §1.21. □

If more structure is available, deeper statements can be made. For this we need

Definition 6.2.5 *A right dual of an object X in a monoidal category $\mathcal{C}$ is an object X^*, equipped with morphisms $ev_X : X^* \otimes X \to 1$ and $coev_X : 1 \to X \otimes X^*$, called the evaluation and coevaluation, satisfying the equalities*

$$X \xrightarrow{coev_X \otimes id_X} (X \otimes X^*) \otimes X \xrightarrow{a_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{id_X \otimes ev_X} X = Id_X$$

$$X^* \xrightarrow{Id_{X^*} \otimes coev_X} X^* \otimes (X \otimes X^*) \xrightarrow{a^{-1}_{X^*,X,X^*}} (X^* \otimes X) \otimes X^* \xrightarrow{ev_X \otimes id_{X^*}} X^* = Id_{X^*}$$

A left dual of X is an object ${}^*X$ that satisfies the analogous property by substituting X for ${}^*X$ and X^* for X in the above definition. These objects are unique up to unique isomorphism and a monoidal category is called rigid if every object has both a left and a right dual.

Definition 6.2.6 *A locally finite k-linear abelian rigid monoidal category $\mathcal{C}$ is a tensor category over k if the bifunctor $\otimes$ is bilinear on morphisms, and* $\mathrm{End}(1) \cong k$.

If H is not only a bialgebra, but also a Hopf algebra, then the abelian monoidal category $\mathrm{Rep}(H)$ also has right duals. The dual of X is the dual vector space X^*, with action $\rho_{X^*}(a) = \rho_X(S(a))^*$, and usual evaluation and coevaluation morphisms coming from Vect. If the antipode is invertible, we also have left duals by taking the dual space of X again, with action $\rho_{{}^*X}(a) = \rho_X(S^{-1}(a))^*$. This sets up the following duality theorem.

Theorem 6.2.7 (Reconstruction theorem for finite dimensional Hopf algebras) *The assignments*

$$(\mathcal{C}, F) \mapsto H = \mathrm{End}(F), \;\; H \mapsto (\mathrm{Rep}(H), \mathrm{Forget})$$

are mutually inverse bijections between

- *Finite tensor categories $\mathcal{C}$ with a fiber functor F, up to monoidal equivalence and isomorphism of monoidal functors,*
- *Finite dimensional Hopf algebras H over k up to isomorphism.*

Proof: This follows from what we just discussed. The antipode on the category is defined by $S(a)_X = a^*_{X^*}$. The proof that this is an antipode is in Etingof [27], Chapter 1, §1.22. □

In reality, one would like to get rid of the finiteness, since this is often too strong a condition. Etingof assumes $\mathcal{C}$ is an essentially small k-linear abelian category, and the functor $F : \mathcal{C} \to \mathrm{Vect}$ is exact and faithful. One then defines

$$\mathrm{Coend}(F) = (\oplus_{X \in \mathcal{C}} FX^* \otimes FX)/E, \tag{6.2}$$

where E is spanned by elements of the form $y_* \otimes F(f)x - F(f)^* y_* \otimes x$, for $x \in FX, y_* \in FY^*, f \in \mathrm{Hom}(X, Y)$. In other words $\mathrm{Coend}(F) = \varinjlim \mathrm{End}(FX)^*$, and we get a coalgebra structure on $\mathrm{Coend}(F)$ (remember that $\mathrm{End}(FX)$ is a finite dimensional algebra). This coalgebra structure can also be constructed directly from $\mathrm{End}(F)$, since it has a pseudocompact topology, see Section 4 of Van den Bergh [77].

Theorem 6.2.8 (Takeuchi's theorem) *With the conventions above, F defines an equivalence between $\mathcal{C}$ and the category of finite dimensional right comodules over* $\mathrm{Coend}(F)$ *(which is also the category of continuous finite dimensional left* $\mathrm{End}(F)$*-modules).*

Proof: See Takeuchi [75]. □

If $\mathcal{C}$ is also monoidal, one can show that $\mathrm{Coend}(F)$ is actually a bialgebra (we will elaborate on this bit in the next subsection). If on top of those things, $\mathcal{C}$ is rigid, we even get a Hopf algebra (with invertible antipode), giving an 'infinite' reconstruction theorem.

Theorem 6.2.9 (Infinite reconstruction theorem for bialgebras and Hopf algebras) *The assignments*

$$(\mathcal{C}, F) \mapsto H = \mathrm{End}(F), \;\; H \mapsto (H - \mathrm{Comod}, \mathrm{Forget})$$

are mutually inverse bijections between

- *k-linear abelian monoidal categories $\mathcal{C}$ with a fiber functor F, up to monoidal equivalence and isomorphism of monoidal functors,*
- *bialgebras over k up to isomorphism.*

and also between

- *tensor categories $\mathcal{C}$ over k with a fiber functor F, up to monoidal equivalence and isomorphism of monoidal functors,*
- *Hopf algebras H over k up to isomorphism.*

Proof: See Etingof [27], Chapter 1, §1.23. □

A taste of abstract expressionism

Even without finiteness, the Tannaka-Krein theory requires abelian categories, which is rather restrictive. This paragraph is a short summary of what is known to us if the category $\mathcal{C}$ is not assumed to be abelian, at least regarding the reconstruction of the bi- or Hopf algebra. The recognition part (when is such and such category the category of representations of such and such structure) is a lot harder and does not seem to have a satisfactory answer at the moment. The most general results seem to be due to Joyal and Street [48]; we will not focus on these, but rather give a presentation based on a category theory course taught by Joost Vercruysse [79] at the Université Libre de Bruxelles. The advantage is that we do not need advanced category theoretic notions, while still being able to say something in a more general setup.

In this section we will work with the category $\mathcal{V} = \mathbf{Vect}$ of all (not necessarily finite dimensional!) vector spaces over a field. Most of the statements can be made for a suitable symmetric monoidal category, and one can check Vercruysse [79] for the exact conditions. We will also not worry about strictness. Suppose $F : \mathcal{C} \to \mathcal{V}$ is a strong, rigid monoidal functor, where $\mathcal{C}$ is rigid monoidal, and the essential image of F is in Vect, the category of finite dimensional vector spaces. The end construction is a generalization of the well known isomorphism for finite dimensional vector spaces $\mathrm{End}(V) \cong V \otimes V^*$. To associate to each object $X \in \mathcal{C}$ the object $FX \otimes FX^*$ in a functorial way, we need cospans.

Definition 6.2.10 *The category* $\mathrm{Cosp}(\mathcal{V})$ *of cospans in* $\mathcal{V}$ *has the same objects as* $\mathcal{V}$*, and a morphism* $\underline{f} : X \to Y$ *is given by* $\underline{f} = (f_0, f_X, f_Y)$*, where* $f_0 \in \mathcal{V}, f_X : X \to f_0, f_Y : Y \to f_0$ *are morphisms in* $\mathcal{V}$*. The identity on an object is* (X, id_X, id_X)*, and composition is given by pushing out:* $(g_0, g_Y, g_Z) \circ (f_0, f_X, f_Y) = (h_0, h_1 \circ f_X, h_2 \circ g_Z)$*, where* (h_0, h_1, h_2) *is the pushout of* (f_Y, g_Y)*.*

Then $\mathcal{V}$ and $\mathcal{V}^{op}$ are embedded in $\mathrm{Cosp}(\mathcal{V})$ via the identity on objects, and by $(f : X \to Y) \mapsto (Y, f, id_Y)$ (respectively $(f : X \leftarrow Y) \mapsto (X, id_X, f)$) on morphisms. The strong monoidal functor F then induces a functor $\hat{F} : \mathcal{C} \to \mathrm{Cosp}(\mathcal{V})$ by $\hat{F}(X) = FX \otimes FX^*$, and $\hat{F}(f) = (FY \otimes FX^*, Ff \otimes id_{FX^*}, id_{FY} \otimes Ff^*)$, for $f : X \to Y$.

Definition 6.2.11 *The end of the functor* $F : \mathcal{C} \to \mathcal{V}$ *is the (category theoretic) limit of the functor* $\hat{F}$*, denoted end*(F)*. The object part of this limit is often denoted* $\int_{X \in \mathcal{C}} FX$*.*

More explicitly, one has

Proposition 6.2.12 *Consider the parallel pair of morphisms* (α, β) *in* $\mathcal{V}$

$$\prod_{X \in \mathcal{C}} FX \otimes FX^* \rightrightarrows \prod_{f : X \to Y} FY \otimes FX^*$$

where α *is induced by the map* $Ff \otimes id_{FX^*}$ *and* β *by* $id_{FY} \otimes Ff^*$*, for* $f : X \to Y$*; then for the object part of the equalizer* (E, e)*, one has* $E \cong \int_{X \in \mathcal{C}} FX$*.*

Proof: This boils down to the fact that the limit of any functor (in our case $\hat{F}$) can be computed as equalizer of a certain parallel pair of morphisms. See Vercruysse [79], Proposition 5.4.5. $\square$

Thus, as an object in $\mathcal{V}$, $\mathrm{end}(F)$ makes the following diagram commute for all f

$$\begin{tikzcd} \mathrm{end}(F) \arrow[r, "e_X"] \arrow[d, "e_Y"] & FX \otimes FX^* \arrow[d, "Ff \otimes id_{FX^*}"] \\ FY \otimes FY^* \arrow[r, "id_{FY} \otimes Ff^*"'] & FY \otimes FX^* \end{tikzcd}$$

where $e_X = \pi_X \circ e$. Dually, one has the coend of F.

Definition 6.2.13 *The coend of $F : \mathcal{C} \to \mathcal{V}$ is the object part in the coequalizer*

$$\coprod_{f:X\to Y} FY^* \otimes FX \rightrightarrows \coprod_{X\in\mathcal{C}} FX^* \otimes FX \to \int^{X\in\mathcal{C}} FX$$

which we write as $\mathrm{coend}(F)$.

This supports the less formal definition Etingof [27] gives (and we used in (6.2)), for $\mathcal{C}$ an essentially small k-linear abelian category.

Proposition 6.2.14 *The end of a functor $F : \mathcal{C} \to \mathcal{V}$ is an algebra. Dually, the coend of such a functor is a coalgebra.*

Proof: One first shows that $\hat{F}$ is a $\mathcal{C}$-algebra in $\mathrm{Cosp}(\mathcal{V})$, meaning that it is an algebra in the category $(\mathrm{Fun}(\mathcal{C}, \mathrm{Cosp}(\mathcal{V})), \otimes, I)$, with pointwise monoidal product. Then, one uses the fact that the object part of the limit of a $\mathcal{C}$-algebra in $\mathrm{Cosp}(\mathcal{V})$ is an algebra in $\mathrm{Cosp}(\mathcal{V})$. Using the equalizer description of $\mathrm{end}(F)$, one can then use its universal property to show that $\mathrm{end}(F)$ is already an algebra in $\mathcal{V}$. The dual statement is analogous. See Joyal and Street [48] or Vercruysse [79] Corollary 5.4.12 for details. □

To show clearly what is special about **Vect**, we phrase the next proposition abstractly.

Proposition 6.2.15 *If for any object $X \in \mathcal{V}$, the endofunctors $- \otimes X$ and $X \otimes -$ preserve limits (colimits), then* $\mathrm{end}(F)$ *is a coalgebra (coend(F) an algebra). In these cases one even gets Hopf algebras.*

Proof: The reason we need the limit preservation property is that then $\mathrm{end}(F \underline{\otimes} F) \cong \mathrm{end}(F) \otimes \mathrm{end}(F)$, where $\otimes$ is the pointwise tensor product, and $\underline{\otimes}$ is a different monoidal structure on functors, where the domain category can be variable. More precisely, given two functors $F : \mathcal{Z} \to \mathcal{V}$ and $F' : \mathcal{Z}' \to \mathcal{V}$, we have $F \underline{\otimes} F' : \mathcal{Z} \times \mathcal{Z}' \to \mathcal{V}$, such that $(F \underline{\otimes} F')(Z, Z') = FZ \otimes F'Z'$ and $(F \underline{\otimes} F')(f, f') = Ff \otimes F'f'$. On morphisms (=natural transformations), $\underline{\otimes}$ is defined in the natural way. The comultiplication on $\mathrm{end}(F)$ is defined as follows: for any two objects $X, Y \in \mathcal{C}$, define $\delta_{X,Y} : \mathrm{end}(F) \to FX \otimes FX^* \otimes FY \otimes FY^*$ by the composition

$$\mathrm{end}(F) \xrightarrow{e_{X\otimes Y}} (FX \otimes FY) \otimes (FX \otimes FY)^* = FX \otimes FY \otimes FY^* \otimes FX^* \xrightarrow{id_{FX} \otimes \gamma_{FY\otimes FY^*, FX^*}} FX \otimes FX^* \otimes FY \otimes FY^*,$$

turning the couple $(\mathrm{end}(F), \delta_{X,Y})$ into a cone on $\hat{F} \underline{\otimes} \hat{F}$. Hence, there is a universal morphism

$$\Delta : \mathrm{end}(F) \to \mathrm{end}(F) \otimes \mathrm{end}(F) \cong \lim \hat{F} \underline{\otimes} \hat{F}$$

The counit is just e_I, where I is the unit in $\mathcal{C}$. The antipode on $\mathrm{end}(F)$ also comes from a suitable cone: for $X \in \mathcal{C}$, define $s_X : \mathrm{end}(F) \to FX \otimes FX^*$ by

$$\mathrm{end}(F) \xrightarrow{e_{X^*}} FX^* \otimes FX \xrightarrow{\gamma_{FX^*, FX}} FX \otimes FX^*$$

The calculations are in Vercruysse [79], Corollary 5.4.15. □

Notice that in our case, $\mathcal{V} = \mathbf{Vect}$, the tensor product functors commute with all colimits, but not necessarily with all limits. It is for this reason that we used the coend in the previous section. Also notice that it is only here that we use the rigidity of $\mathcal{C}$; if one only cares about bialgebras, being monoidal is sufficient. These are all the ingredients necessary for the Tannaka-Krein reconstruction.

Theorem 6.2.16 *For any Hopf algebra H, there are canonical morphisms $\alpha : H \to \mathrm{end}(F)$ and $\beta : \mathrm{coend}(F) \to H$, where F is the (respective) usual forgetful functor. For $\mathcal{V} = \mathbf{Vect}$, β is an isomorphism.*

Proof: See Joyal and Street [48]. We just remark that in the proof of β being an isomorphism, the fundamental theorem of coalgebras plays a crucial role. □

Bridging the gap

We are now in the position to establish a link between Tannaka-Krein and $\underline{\mathrm{end}}(A)$, for A a N-homogenous algebra. The following subsections are 'work in progress', and should be considered as an informal collection of ideas. Just like in ring theory, there exists a notion of free monoidal category on a given set of objects $X_1, \ldots, X_n$. If, in addition, there are given a number of morphisms $\phi_1, \ldots, \phi_m$, then one can also the consider the free monoidal category on the objects and the morphisms. This is exactly what one would expect, though the rigorous construction is a bit technical, and we refer to Maclane [54]. Informally, one adds an identity object, the products of all objects, identity morphisms for every object, the morphisms required for the unit property, the associators, the products of all morphisms, and all possible compositions, possibly repeating some steps. One also imposes the conditions required in the definition of a monoidal category. Since we will be considering 'small' strict monoidal categories, all will be clear. Define $\mathcal{C}$ to be the free k-linear monoidal category on two objects v and r, and one morphism $r \to v^{\otimes N}$. Given an N-homogeneous algebra $A = TV/(R)$, for $R \subset V^{\otimes N}$, define a strong monoidal functor $F : \mathcal{C} \to \mathrm{Vect}$ by demanding that $F(v) = V, F(r) = R$, and $F(r \to v^{\otimes N}) = R \hookrightarrow V^{\otimes N}$.

Proposition 6.2.17 *For $A, \mathcal{C}$ and F as defined above, we can calculate the bialgebra $\mathrm{coend}(F)$ (remember that this only needs that $\mathcal{C}$ and F are (strong) monoidal). To A, one can also associate the bialgebra $\underline{\mathrm{end}}(A)$ introduced in Chapter 4. As bialgebras*

$$\underline{\mathrm{end}}(A) \cong \mathrm{coend}(F).$$

Proof: As we saw in Section 4.1,

$$\underline{\mathrm{end}}(A) = T(V^* \otimes V)/(\pi_N(R^\perp \otimes R)).$$

For $\mathrm{coend}(F)$, we use the formula

$$\mathrm{coend}(F) = (\oplus_{X \in \mathcal{C}} FX^* \otimes FX)/E.$$

Now the direct sum in the numerator is immediately seen to be equal to $T(V^* \otimes V)$, since every power of v is an object in $\mathcal{C}$, and the other products are all subspaces of some $V^{\otimes n}$. Remember that the subspace E is spanned by elements $e = y_* \otimes F(f)x - F(f)^* y_* \otimes x$. In our case, it suffices to look at $f : r \to v^{\otimes N}$, since all morphisms are built from this one. For $y_* : V^{\otimes N} \to k$, and $x \in R$, we have $F(f) : R \hookrightarrow V^{\otimes N}$, and thus

$$e = y_* \otimes x - y_*|_R \otimes x.$$

Since $R^\perp = \{r \in V^* \otimes \cdots \otimes V^* \mid r(R) = 0\}$, the statement is clear. $\square$

Thinking in reverse, if we start with $\underline{\mathrm{end}}(A)$, and define $\widetilde{F}$ to be the forgetful functor $\widetilde{F} : \mathrm{CoMod}(\underline{\mathrm{end}}(A)) \to \mathrm{Vect}$, we know from the previous section that $\mathrm{coend}(\widetilde{F}) \cong \underline{\mathrm{end}}(A)$ as bialgebras. Like we said before, a Tannaka-type theorem relating $\mathrm{CoMod}(\underline{\mathrm{end}}(A))$ and $\mathcal{C}$ does not seem to exist. The fact that the coend's of the functors F and $\widetilde{F}$ are isomorphic suggest that the couple $(\mathcal{C}, F)$ is important in understanding $\underline{\mathrm{end}}(A)$. Furthermore, there is an obvious functor

$$SW : \mathcal{C} \to \mathrm{CoMod}(\mathrm{coend}(F)) : A \mapsto FA.$$

Coming from representation theory, one is inclined to call the couple $(\mathcal{C}, F)$ a 'tensor quiver representation': there is a vertex for every object, and arrows are allowed to go between tensor products of vertices. Just like for regular quivers, a representation is an assignment of vector space to each vertex, and a linear map for each arrow. Now the category $\mathcal{C}$ is in a certain sense graded: $\mathcal{C} = \oplus_{n\in\mathbb{N}}\mathcal{C}^n$, and each $\mathcal{C}^n$ is a full subcategory, such that $\mathcal{C}^n \otimes \mathcal{C}^m \to \mathcal{C}^{n+m}$. This is accomplished by giving v degree 1, and r degree N. From the computations in Proposition 6.2.17, $\mathrm{coend}(F) = \oplus_n \mathrm{coend}(F_n)$, where F_n is the restriction of F to $\mathcal{C}^n$. Each $\mathcal{C}^n$ now has a finite number of indecomposable objects, these are just the products of v and r of total degree n. Thus, $\mathrm{coend}(F_n)$ is dual to the algebra $\mathrm{end}(F_n)$, which is exactly $Z_n(A)$, as defined in Definition 4.2.4, by a computation completely similar to the previous proposition. This bridges the gap between the two viewpoints: to understand the representation theory of $\mathrm{coend}(F)$, it suffices to understand the representations of $\mathrm{end}(F_n)$ for every n. It seems to be combinatorially easier to work with $\mathcal{C}$ and F than to work directly with the $Z_n(A)$.

A small example to illustrate this: consider $A = TV$, for $\dim(V) = m$, the free algebra on m generators. To study $\underline{\mathrm{end}}(TV)$, we define $\mathcal{C}$ to be the free monoidal, k-linear category on one object v, and no morphisms. The fiber functor F just sends v to V. There is now only one indecomposable object in every degree, $v^{\otimes n}$, so $\mathrm{end}(F_n) = \mathrm{End}(V^{\otimes n})$. This is just a matrix algebra, so the representation theory of the full bialgebra $\underline{\mathrm{end}}(TV)$ is semisimple, with one simple in each degree. The character ring is then just $\mathbb{Z}[S]$, where S corresponds to the comodule V itself.

Schur-Weyl revisited

Consider a strict monoidal category $\mathcal{C}$ and a strong monoidal functor $F : \mathcal{C} \to \mathrm{Vect}$. By using the universal property of the coend,

$$SW : \mathcal{C} \to \mathrm{CoMod}(\mathrm{coend}(F)) : A \mapsto FA$$

is a properly defined functor.

Definition 6.2.18 *The couple $(\mathcal{C}, F)$ is said to satisfy Schur-Weyl duality if the functor SW is full.*

Notice that Tannaka-Krein duality is essentially the statement that this functor is an equivalence (see Theorem 6.2.8), so Schur-Weyl duality can be viewed as a weaker version of Tannaka-Krein duality. To motivate the name, let's look at the classical case: consider $\mathcal{C}$ the symmetric strict k-linear monoidal category on 1 object v. Define F by $F(v) = V \cong k^n$. Then $\mathrm{coend}(F) = \mathcal{O}(M_n)$, the classical coordinate ring of the monoid of $n \times n$ matrices: this is due to the fact that we also demanded the category to be symmetric. Remember from Theorem 5.2.10 that classical Schur-Weyl duality tells us that

$$kS_r \to \mathrm{End}^{\mathcal{O}(M_n)}(V^{\otimes r})$$

is surjective for every r. Since $\mathrm{End}_{\mathcal{C}}(v^{\otimes r}) = kS_r$, again because we're working in a symmetric monoidal category, we see that the couple $(\mathcal{C}, F)$ satisfies Schur-Weyl duality exactly because classical Schur-Weyl duality holds.

In the case of a distributive algebra $A = TV/(R)$, take $\mathcal{C} = \oplus\mathcal{C}^n$, where $\mathcal{C}^n = Q_{A,n-N+1}$, the quivers defined in Corollary 4.3.12. A quiver can be thought of as a category by considering the free category on the quiver, with vertices as objects, arrows as morphisms, and by adding the necessary identity morphisms. The monoidal structure is the same as in Chapter 3. We can analyse this situation locally, for each $\mathcal{C}^n$ separately. Take $F^n : \mathcal{C}^n \to \mathrm{Vect}$ to be the quiver representation that sends each vertex to the corresponding R_I^n, and the arrows to the obvious inclusions. Then $\mathrm{coend}(\oplus_n F^n)$ is seen to be $\underline{\mathrm{end}}(A)$. Asking for fullness of the functor

$$SW : \oplus_n \mathcal{C}_n \to \mathrm{CoMod}(\underline{\mathrm{end}}(A))$$

then leads in a natural way to the main theorems of Kriegk and Van den Bergh [51], which we saw as Theorem 4.3.13 and Corollary 4.3.10. Schur-Weyl duality was essentially due to the fact that the basis from Proposition 3.1.3 guarantees that the functor

$$\widetilde{F^n} : \mathrm{Cube}_{n-N+1} \to \mathrm{Vect} : P_I^{n-N+1} \mapsto F^n(P_I^{n-N+1})$$

is exact. It seems interesting to characterize the settings $(\mathcal{C}, F)$ satisfying Schur-Weyl duality.

6.3 The Hopf algebra $\mathcal{O}_{nc}(GL_2)$ and its representations

In this section, we will try to deduce the representation theory of the Hopf algebra version of $\mathcal{O}_{nc}(M_n)$ in the simplest case, that of $n = 2$. This corresponds to the algebra $A = k[x, y]$. To construct this Hopf algebra, it is again instructive to view the generators of $\mathcal{O}_{nc}(M_2)$ as a matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Remember that the relations are given by 'column commutativity' and the 'determinant identity'; explicitly:

$$ac - ca = 0,\ bd - db = 0,\ ad - cb = da - bc.$$

First off, we add a variable δ, that is supposed to represent the determinant, so we add the equations

$$\delta = ad - cb = da - bc,$$

Then, we add another variable δ^{-1}, such that

$$\delta\delta^{-1} = 1 = \delta^{-1}\delta$$

With these equations, we see that the formal matrix multiplication equation

$$\begin{pmatrix} \delta^{-1}d & -\delta^{-1}b \\ -\delta^{-1}c & \delta^{-1}a \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

holds. To get a Hopf algebra, it will be necessary to make sure the generator matrix also has a right inverse, so we impose the four extra relations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \delta^{-1}d & -\delta^{-1}b \\ -\delta^{-1}c & \delta^{-1}a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now define $\mathcal{O}_{nc}(GL_2)$ to the the algebra $k\langle a, b, c, d, \delta, \delta^{-1}\rangle/I$, where I is the ideal generated by the ten relations above.

Proposition 6.3.1 *The algebra $\mathcal{O}_{nc}(GL_2)$ is a Hopf algebra, with*

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ \Delta(\delta) = \delta \otimes \delta, \ \Delta(\delta^{-1}) = \delta^{-1} \otimes \delta^{-1},$$

$$S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \delta^{-1}d & -\delta^{-1}b \\ -\delta^{-1}c & \delta^{-1}a \end{pmatrix}, \ S(\delta) = \delta^{-1}, \ S(\delta^{-1}) = \delta,$$

$$\epsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \epsilon(\delta) = \epsilon(\delta^{-1}) = 1.$$

Proof: All structure maps are defined on the generators, and we use this to force Δ and ϵ to be algebra morphisms. To check if $\mathcal{O}_{nc}(GL_2)$ is a bialgebra, it suffices to check this doesn't contradict the imposed relations (coassociativity and counitality are obvious). First the comultiplication:

$$\begin{aligned}
\delta \otimes \delta = \Delta(\delta) = \Delta(ad - cb) &= (a \otimes a + b \otimes c)(c \otimes b + d \otimes d) - (c \otimes a + d \otimes c)(a \otimes b + b \otimes d) \\
&= ac \otimes ab + ad \otimes ad + bc \otimes cb + bd \otimes cd - ca \otimes ab - cb \otimes ad \\
&- da \otimes cb - db \otimes cd \\
&= (ac - ca) \otimes ab + (ad - cb) \otimes (ad - cb) + (bd - db) \otimes cd
\end{aligned}$$

$$\Delta(\delta\delta^{-1}) = (\delta \otimes \delta)(\delta^{-1} \otimes \delta^{-1}) = 1 \otimes 1,$$

$$\begin{aligned}
1 \otimes 1 = \Delta(1) &= \Delta(a\delta^{-1}d - b\delta^{-1}c) \\
&= (a \otimes a + b \otimes c)(\delta^{-1} \otimes \delta^{-1})(c \otimes b + d \otimes d) \\
&- (a \otimes b + b \otimes d)(\delta^{-1} \otimes \delta^{-1})(c \otimes a + d \otimes c) \\
&= a\delta^{-1}c \otimes a\delta^{-1}b + a\delta^{-1}d \otimes a\delta^{-1}d + b\delta^{-1}c \otimes c\delta^{-1}b + b\delta^{-1}d \otimes c\delta^{-1}d \\
&- a\delta^{-1}c \otimes b\delta^{-1}a - a\delta^{-1}d \otimes b\delta^{-1}c - b\delta^{-1}c \otimes d\delta^{-1}a - b\delta^{-1}d \otimes d\delta^{-1}c \\
&= a\delta^{-1}c \otimes (a\delta^{-1}b - b\delta^{-1}a) + b\delta^{-1}d \otimes (c\delta^{-1}d - d\delta^{-1}c) \\
&+ (a\delta^{-1}d - b\delta^{-1}c) \otimes (a\delta^{-1}d - b\delta^{-1}c) \\
&= 1 \otimes 1
\end{aligned}$$

The other calculations are similar. For the counit, the compatibility is immediately obvious. Now S is defined by demanding it to be an anti-algebra homomorphism. First we check that it is well defined:

$$\begin{aligned}
0 = S(0) = S(ac - ca) &= -\delta^{-1}c\delta^{-1}d + \delta^{-1}d\delta^{-1}c = \delta^{-1}(d\delta^{-1}c - c\delta^{-1}d) \\
S(ad - cb) &= \delta^{-1}a\delta^{-1}d - \delta^{-1}b\delta^{-1}c = \delta^{-1}(a\delta^{-1}d - b\delta^{-1}c) = \delta^{-1} \\
S(a\delta^{-1}d - b\delta^{-1}c) &= \delta^{-1}a\delta\delta^{-1}d - \delta^{-1}c\delta\delta^{-1}b = \delta^{-1}(ad - cb) = 1,
\end{aligned}$$

and all other computations are completely similar. Remains to verify the antipode axiom $S(c_{(1)})c_{(2)} = \epsilon(c) = c_{(1)}S(c_{(2)})$. This can be checked on almost all generators simultaneously from the formal matrix resemblance:

$$\begin{aligned} S(M_{(1)})M_{(2)} &= \begin{pmatrix} \delta^{-1}d & -\delta^{-1}b \\ -\delta^{-1}c & \delta^{-1}a \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \delta^{-1}d & -\delta^{-1}b \\ -\delta^{-1}c & \delta^{-1}a \end{pmatrix} = M_{(1)}S(M_{(2)}), \end{aligned}$$

and it obviously holds for δ and δ^{-1} as well. □

Representations: the naive approach

From now on we will always consider $\mathcal{O}_{nc}(GL_2)$ with its associated Hopf algebra structure. We would like to understand the representation theory of this Hopf algebra, just like we did for $\mathcal{O}_{nc}(M_n)$. This did not turn out to be easy, and both the approach via non-commutative symmetric functions, and the approach via the generalized Tannaka-Krein formalism have not yielded any effective results. Two methods will be introduced, one computational, and one based on the representation theory of algebraic groups, see Jantzen [47]. As a small interlude, we will highlight the connection with so called Manin matrices, though we did not succeed in applying this theory in our setting.

The first approach draws upon the Bergman diamond lemma, which is a tool that comes in handy when trying to prove that a set of words in a finitely presented algebra forms a basis. It is a non-commutative version of the theory of Gröbner bases in commutative algebra. For a free algebra $k\langle x_1, \ldots, x_n\rangle$, and an ordering on the variables, say $x_1 < \cdots < x_n$ (though it is often convenient to pick another ordering), we get an ordering on all words by considering the degree lex ordering. Given an element of the free algebra, its leading word is the largest monomial with respect to this ordering. For example, in the polynomial $xyx + x^2y + x^4$, the order of the monomials is $x^2y < xyx < x^4$, and x^4 is thus the leading word. Given a set of generators $f_1, \ldots, f_r$ of some ideal $I \lhd k\langle x_1, \ldots, x_n\rangle$, and $w_1, \ldots, w_r$ are the leading (monic, different) words in the f_i. A word is called reduced if it does not contain any w_i; if it does, use f_i to rewrite the word as a linear combination of smaller words until we only have (linear combinations of) reduced words left. The images of the reduced words in $k\langle x_1, \ldots, x_n\rangle/I$ are then a k-spanning set. If they also form a basis, we say the f_i form a non-commutative Gröbner basis for I. The diamond lemma provides a way of verifying this, using a (possibly non-terminating) algorithm to obtain such a basis. To state the lemma, we need two more concepts: if a word w can be written both as $w = w_iu$ and as $w = vw_j$, such that w_i and w_j overlap in at least one letter, this is called an overlap ambiguity, and one has different ways of reducing w. If the different ways coincide, and we eventually get the same linear combination of reduced words, one says the ambiguity is resolvable. If not, and the two different linear combinations are r_1 and r_2, then we add $r_1 - r_2$ to the f_i and try again. An inclusion ambiguity happens when $w = w_i$ and $w = tw_ju$, so w is both a leading word, but also contains another. The same terminology applies to this kind of ambiguity. The Bergman diamond lemma then says

Lemma 6.3.2 (Bergman diamond lemma) *Let (a possibly infinite number of) elements $f_1, f_2, \ldots$ in $k\langle x_1, \ldots, x_n\rangle$ generate an ideal I, with distinct leading words $w_1, w_2, \ldots$. Then the f_i are a Gröbner basis for I if and only if all overlap and inclusion ambiguities among the w_i resolve.*

For a proof, see Bergman [13]. We will immediately apply this to $\mathcal{O}_{nc}(GL_2)$.

Proposition 6.3.3 *The Hopf algebra $\mathcal{O}_{nc}(GL_2)$ has a basis of the form*

$$\delta^{a_0} w_1 \delta^{a_1} w_2 \dots w_n \delta^{a_n},$$

where $a_i \in \mathbb{Z}$, $a_i \neq 0$ for $i \notin \{0, n\}$, and the w_i are non-empty words in the symbols a, b, c, d with non-decreasing row index. If $a_i = -1$, and $i \notin \{0, n\}$ then the column index of the symbol on the left and on the right of $\delta^{a_i} = \delta^{-1}$ should be non-decreasing as well.

Proof: Writing the equations of $\mathcal{O}_{nc}(GL_2)$ in the following way:

$$\begin{array}{lll} da - bc = \delta, & ad - cb = \delta, & ca = ac, \\ db = bd, & a\delta^{-1}d - b\delta^{-1}c = 1, & -a\delta^{-1}b + b\delta^{-1}a = 0, \\ c\delta^{-1}d - d\delta^{-1}c = 0, & -c\delta^{-1}b + d\delta^{-1}a = 1, & \delta \cdot \delta^{-1} = 1, \\ \delta \cdot \delta^{-1} = 1 & & \end{array}$$

and using the ordering $\delta^{-1} < \delta < a < b < c < d$, the dominant terms are

$$\begin{array}{llll} da, & cb, & ca, & db, \\ b\delta^{-1}c, & b\delta^{-1}a, & d\delta^{-1}c, & d\delta^{-1}a \\ \delta \cdot \delta^{-1}, & \delta^{-1} \cdot \delta. & & \end{array}$$

From these, we can see that the overlaps are given by

$$cb\delta^{-1}c,\ cb\delta^{-1}a,\ b\delta^{-1}cb,\ d\delta^{-1}cb,\ b\delta^{-1}ca,\ d\delta^{-1}ca,\ db\delta^{-1}c,\ db\delta^{-1}a.$$

We have checked by hand that all ambiguities resolve. To illustrate the calculations, we will do one example here:

$$d\delta^{-1}(cb) = d\delta^{-1}(ad - \delta) = (1 + c\delta^{-1}b)d - d = c\delta^{-1}bd,$$

while

$$(d\delta^{-1}c)b = c\delta^{-1}db = c\delta^{-1}bd,$$

so we see this overlap ambiguity resolves. The others are completely similar. From Bergman's diamond lemma, we deduce that the reduced words form a k-basis of $\mathcal{O}_{nc}(GL_2)$. Looking at the dominant terms, the first row tells us that subwords containing no δ^a have to have non-decreasing row index. The second set of four tells us that if $x\delta^{-1}y$ is a reduced word, with $x, y \in \{a, b, c, d\}$, then the column index of x with respect to y is non-decreasing. □

Now, let us introduce the zeroth Hochschild homology of a coalgebra C:

$$\mathrm{HH}_0(C) = \{c \in C \mid c_{(1)} \otimes c_{(2)} = c_{(2)} \otimes c_{(1)}\}.$$

This is exactly the cocenter of C. It should be remarked that this fits into a general theory of Hochschild (co)homology for coalgebras, developed by Doi and others in [22, 29]. As a first step towards computing the representation ring of $\mathcal{O}_{nc}(GL_2)$, we will compute $\mathrm{HH}_0(\mathcal{O}_{nc}(GL_2))$.

Proposition 6.3.4 *The zeroth Hochschild homology of $\mathcal{O}_{nc}(GL_2)$ is isomorphic as a k-space to $k\langle \Lambda_1, \Lambda_2 \rangle_{\Lambda_2}$, a localized free algebra on 2 generators, via a map*

$$\Phi : k\langle \Lambda_1, \Lambda_2 \rangle_{\Lambda_2} \to \mathrm{HH}_0(\mathcal{O}_{nc}(GL_2)).$$

Proof: From now on, we will use the notation V, R, and R^{-1} as shorthand for the natural simple comodules

$$kx + ky = V \to \mathcal{O}_{nc}(GL_2) \otimes V : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix}$$

$$k(xy - yx) = R \to \mathcal{O}_{nc}(GL_2) \otimes R : (xy - yx) \mapsto \delta \otimes (xy - yx)$$

$$k(x^*y^* + y^*x^*) = R^{-1} \to \mathcal{O}_{nc}(GL_2) \otimes R^{-1} : (x^*y^* + y^*x^*) \mapsto \delta^{-1} \otimes (x^*y^* + y^*x^*)$$

Taking our cue from the commutative theory again, the elements Λ_1, Λ_2 and Λ_2^{-1} should correspond to V, R and R^{-1}. One can then use a character map $\chi : \mathcal{R}(U) \to U$ in the comodule sense, where U is a bialgebra. This is given by $\chi_V = \mathrm{Tr}_V \delta$, for a comodule $\delta : V \to U \otimes V$: if $(v_i)_i$ is a basis of V, then $\mathrm{Tr}_V \delta = \sum_i v_{i[1]} \otimes v_i^*(v_{i[2]}) \in U$. It is rather immediate that $\chi_{V \oplus V'} = \chi_V + \chi_{V'}$, and $\chi_{V \otimes V'} = \chi_V \cdot \chi_{V'}$, so the map is a ring morphism. In our case, it is easy to see that χ has the effect

$$V \mapsto a + d, \;\; R \mapsto \delta, \;\; R^{-1} \mapsto \delta^{-1}.$$

Defining Φ to be the composition, we get a map that lands in $\mathrm{HH}_0(\mathcal{O}_{nc}(GL_2))$:

$$\Phi : k\langle \Lambda_1, \Lambda_2 \rangle_{\Lambda_2} \to \mathrm{HH}_0(\mathcal{O}_{nc}(GL_2)).$$

Using the freshly computed Gröbner basis of $\mathcal{O}_{nc}(GL_2)$, we will show this map is an isomorphism of k-spaces. Given a monomial $m = \Lambda_2^{a_0} \Lambda_1^{b_0} \Lambda_2^{a_1} \cdots \Lambda_1^{b_{n-1}} \Lambda_2^{a_n}$, the dominating term in $\Phi(m)$ is the basis element $\delta^{a_0} d^{b_0} \delta^{a_1} \cdots d^{b_{n-1}} \delta^{a_n}$. It is furthermore obvious by looking at the relations of $\mathcal{O}_{nc}(GL_2)$ as described in the previous proposition, that this element can only come from m. Since the image of every monomial contains a unique basis element, Φ has to be injective. For surjectivity, we claim that the following identities hold

$$\mathrm{dom}\Big(\Delta\big(\prod_i x_i\big)\Big) = \prod_i \mathrm{dom}(\Delta x_i), \tag{6.3}$$

where $x_i \in \{a, b, c, d, \delta, \delta^{-1}\}$, $\prod_i x_i$ is a reduced monomial, and dom takes the dominating term. Furthermore,

$$\mathrm{dom}(\Delta(\alpha + \beta)) = \mathrm{dom}(\Delta \alpha), \tag{6.4}$$

where $\alpha > \beta$ are reduced words. The order we put on $\mathcal{O}_{nc}(GL_2) \otimes \mathcal{O}_{nc}(GL_2)$ is the degree lex order (eg. from left to right). For future reference, we table the generators, their coproducts and the dominating terms, and the relations (in the form dominating term $= \cdots$).

Generator	Coproduct	Dominating term
a	$a \otimes a + b \otimes c$	$b \otimes c$
b	$a \otimes b + b \otimes d$	$b \otimes d$
c	$c \otimes a + d \otimes c$	$d \otimes c$
d	$c \otimes b + d \otimes d$	$d \otimes d$
δ	$\delta \otimes \delta$	$\delta \otimes \delta$
δ^{-1}	$\delta^{-1} \otimes \delta^{-1}$	$\delta^{-1} \otimes \delta^{-1}$

Relations		
da	$=$	$\delta + bc$
cb	$=$	$ad - \delta$
ca	$=$	ac
db	$=$	bd
$b\delta^{-1}c$	$=$	$a\delta^{-1}d - 1$
$b\delta^{-1}a$	$=$	$a\delta^{-1}b$
$d\delta^{-1}c$	$=$	$c\delta^{-1}d$
$d\delta^{-1}a$	$=$	$c\delta^{-1}b + 1$
$\delta \cdot \delta^{-1}$	$=$	1
$\delta^{-1} \cdot \delta$	$=$	1

Assuming the formulas hold, take $w \in \mathrm{HH}_0(\mathcal{O}_{nc}(GL_2))$. By (6.3) and (6.4), $\mathrm{dom}(\Delta(w))$ is a product of dominating terms of the generators, in other words:

$$\mathrm{dom}(\Delta(w)) = (b \otimes c)^{i_1} \cdot (b \otimes d)^{j_1} \cdot (d \otimes c)^{k_1} \cdot (d \otimes d)^{l_1} \cdot (\delta \otimes \delta)^{m_1} \cdot (b \otimes c)^{i_2} \cdots ,$$

where only the m_i can attain negative values. Notice that the only possible reductions that can take place among these products are those coming from $d\delta^{-1}c = c\delta^{-1}d$. Now since w is in the cocenter, two situations can occur. Either $\mathrm{dom}(\Delta(w))$ is symmetric (for example: $ad \otimes ad$), or it isn't. In the first case, from the formula above, we see that

$$\mathrm{dom}(\Delta(w)) = (d \otimes d)^{a_1} (\delta \otimes \delta)^{b_1} (d \otimes d)^{a_2} \cdots .$$

Looking back at the two tables, it is clear that the only way a dominating term of this kind can appear in a cocentral word is if it is a sum of products of $a + d, \delta$ and δ^{-1}. The other case is that $\mathrm{dom}(\Delta(w))$ is not symmetric, but then we know that its 'mirror' also has to appear in w (eg. if the dominating term is $a \otimes b$, then w also contains $b \otimes a$). Using (6.3) again, this means that $\Delta(w)$ contains a term of the form

$$(c \otimes b)^{i_1} \cdot (d \otimes b)^{j_1} \cdot (c \otimes d)^{k_1} \cdot (d \otimes d)^{l_1} \cdot (\delta \otimes \delta)^{m_1} \cdot (c \otimes b)^{i_2} \cdots .$$

Looking back at the tables again, this seems only to be possible if w is a sum of products of d, δ and δ^{-1}: the coproducts of b and c also seem to give the right terms, but upon inspecting the relations, the a's that appear cannot vanish. Again, w is in the cocenter, and since powers of d also provide $c \otimes b$ to some power, one has to correct by looking at powers of $a + d$, and we get the same result, proving the proposition. Remains to check that (6.3) and (6.4) hold. This is due to the fact that in the coproduct of a generator, the dominating term strictly dominates the smaller term: in both factors of the tensor product there is a strict inequality. For example, $\Delta(b) = a \otimes b + b \otimes d$, and $a < b, b < d$. This can be used to verify (6.3), since no non-trivial reductions take place in products of dominating terms. For α and β linear combinations of reduced monomials, formula (6.4) holds because Δ preserves the order. □

To establish a link with the representation ring of $\mathcal{O}_{nc}(GL_2)$, we need some facts about coalgebra representation theory, see Chin [19] for example.

Definition 6.3.5 *The coradical C_0 of C is defined to be the sum of the simple subcoalgebras of C. This is also the socle of C as a comodule, on either side. That is, it is the sum of the simple subcomodules of ${}^C C$, or of C^C.*

Simple modules of a ring R correspond to simple modules of the reduced ring $R/\mathrm{rad}(R)$, $\mathrm{rad}(R)$ denoting the Jacobson radical. In our setting we have

Lemma 6.3.6 *The simple comodules of C are the same as those of C_0.*

Proof: If M is a simple C_0-comodule, then M is a simple C-comodule via

$$M \to C_0 \otimes M \to C \otimes M.$$

The other way round, if (N, ρ) is a simple C-comodule, and $(n_i)_i$ is a basis of N, then

$$\rho(n_i) = \sum_j c_{ij} \otimes n_j,$$

for uniquely determined $c_{ij} \in C$. Now coassociativity implies that

$$\rho(c_{ij}) = \sum_k c_{ik} \otimes c_{kj},$$

and denoting the span of the c_{ij} by $cf(N)$, we see that $cf(N)$ is a subcoalgebra of C. In our case, N is simple, so $cf(N)$ is a simple subcoalgebra of C, and thus, $cf(N) \subset C_0$, so we see that N is a simple C_0-comodule. $\square$

This fits into a coalgebraic version of the Artin-Wedderbun theorem, see Green [40]. Now we can prove

Lemma 6.3.7 *For k a field of characteristic 0, and a k-coalgebra C, the character map*

$$k \otimes_{\mathbb{Z}} \mathrm{Rep}(C) \to \mathrm{HH}_0(C)$$

is injective.

Proof: By the previous argument, we have $\mathrm{Rep}(C) = \mathrm{Rep}(C_0)$. Obviously, $\mathrm{HH}_0(C_0) \to \mathrm{HH}_0(C)$ is injective, so we can restrict to C being cosemisimple: a direct sum of simple subcoalgebras, and so even to C being simple. The fundamental theorem of coalgebras, see Theorem 4.2.2, says that C is finite dimensional, so in particular, $A = C^*$ is a finite dimensional simple k-algebra. The statement is proven if the map

$$\phi : k \otimes_{\mathbb{Z}} \mathrm{Rep}(A) \to (A/[A,A])^* : M \mapsto (\bar{a} \mapsto \mathrm{Tr}_M(a))$$

is injective. Now $\mathrm{Rep}(A) = \mathbb{Z}$, and the map is k-linear, so it suffices to observe that $\phi(M)(1) = \dim(M)$ is non-zero. $\square$

Proposition 6.3.8 *One has*

$$k \otimes_{\mathbb{Z}} \mathrm{Rep}(\mathcal{O}_{nc}(GL_2)) \cong k\langle \Lambda_1, \Lambda_2 \rangle_{\Lambda_2}$$

Proof: From Proposition 6.3.4, we have

$$\mathrm{HH}_0(\mathcal{O}_{nc}(GL_2)) = k\langle \Lambda_1, \Lambda_2 \rangle_{\Lambda_2}.$$

Now Λ_1 and Λ_2 are in the essential image of the character map, so the induced map

$$k \otimes_{\mathbb{Z}} \mathrm{Rep}(\mathcal{O}_{nc}(GL_2)) \to \mathrm{HH}_0(\mathcal{O}_{nc}(GL_2))$$

is surjective. The statement follows after combining it with the previous lemma. $\square$

It is unclear to us at the moment whether a descent argument can be made to conclude that $\mathrm{Rep}(\mathcal{O}_{nc}(GL_2)) = \mathbb{Z}\langle \Lambda_1, \Lambda_2 \rangle_{\Lambda_2}$.

Representations: the 'less' naive approach

The above proof should make it clear that the (rather clumsy) methods used in Proposition 6.3.4 are by no means generalizable to the $n > 2$ setting, since they depend on a direct manipulation of the algebra generators and relations. A slightly more conceptual approach, though still rather algorithmic in nature, uses the specific structure of the relations and is based on the concept of a Manin matrix . Manin's article [55] was taken up by some Russian mathematical physicists, among them Chervov, Rubtsov et al. [16, 17], who found in the article the right framework to study certain (quantum) integrable systems, where matrices with non-commutative entries appear naturally.

Definition 6.3.9 *Let M be an $n \times m$ matrix with elements M_{ij} in a (not necessarily commutative) ring R. This matrix is called a Manin matrix if*

- *elements in the same column commute*
- *commutators of cross terms of 2×2 submatrices are equal:*

$$[M_{ij}, M_{k,l}] = [M_{kj}, M_{il}]$$

It is immediate that the matrix of generators of $\mathcal{O}_{nc}(M_n)$ satisfies this definition. In [16], the authors argue that determinantal identities hold true for Manin matrices in a form identical to the commutative case. In particular, they are able to show exact analogues of Cramer's inversion formula, the Capelli identity, the Cayley-Hamilton theorem, Jacobi's ratio theorem, Sylvester's theorem, and many more. From a ring theory/abstract algebra viewpoint, generic matrices with non-commutative entries have been studied before, but in these works no satisfactory theory of determinants exists (see Dieudonné [21] for a well known attempt). This is not true for Manin matrices, providing an interesting setting for ring theorist to study linear algebra. To at least give some idea of why exactly these relations show up, we give some simple computations in the $n = 2$ case. Suppose

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a matrix with entries in a non-commutative ring R. Then any one of the following theorems hold if and only if M is a Manin matrix:

- Cramer's rule:

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} da - bc & db - bd \\ -ca + ac & -cb + ad \end{pmatrix} = \begin{pmatrix} ad - cb & 0 \\ 0 & ad - cb \end{pmatrix}.$$

- Cayley-Hamilton:

$$\begin{aligned} M^2 - (a+d)M + (ad - cb)\mathrm{Id} &= \begin{pmatrix} a^2 + bc & ab + bd \\ ca + dc & cb + d^2 \end{pmatrix} - \begin{pmatrix} a^2 + da & ab + db \\ ac + dc & ad + d^2 \end{pmatrix} + \begin{pmatrix} ad - cb & 0 \\ 0 & ad - cb \end{pmatrix} \\ &= \begin{pmatrix} (bc - da) + (ad - cb) & bd - db \\ ca - ac & 0 \end{pmatrix} = \begin{pmatrix} [a,d] - [c,b] & [b,d] \\ [c,a] & 0 \end{pmatrix}. \end{aligned}$$

- Binet: assume all elements of M commute with all elements of N, then

$$\begin{aligned} det(MN) - det(M)det(N) =& [M_{11}, M_{21}]N_{11}N_{12} + [M_{12}, M_{22}]N_{21}N_{22} \\ &+ ([M_{11}, M_{22}] - [M_{21}, M_{12}])N_{21}N_{12}. \end{aligned}$$

In higher dimensions, these properties remain valid, see [16], though they do not in general characterize Manin matrices. A word of caution is in order: though at first these results seem rather surprising, a moment's reflection shows that this is actually very much expected. In this theory, all determinants have to be calculated using a column expansion; if one would also like the row-version, the 'other half' of the relations would have to be added, and the minimal relations required for their being equal would give us usual matrix rings. What this theory essentially does, is it decouples the row-column symmetry in linear algebra, and sees what's left.

Remark 6.3.10 *We have tried applying this theory to deduce minimal relations necessary to turn $\mathcal{O}_{nc}(M_n)$ into the Hopf algebra $\mathcal{O}_{nc}(GL_n)$, but this turned out to be rather tricky.*

Representations: the right approach?

A much more conceptual approach to the representations of $\mathcal{O}_{nc}(GL_2)$ is based on the representation theory of algebraic groups in characteristic p, which is fully developed in Jantzen [47]. In the commutative setting one would consider GL_2, which is a reductive group, the algebraic analogue of a compact Lie group. The theory of dominant weights used to study the representations of compact Lie groups carries over to this setting. For a G-representation V, one considers the restriction to a Borel subgroup: a maximal, Zariski closed and solvable subgroup. By the Lie-Kolchin theorem, V contains a B-eigenvector v. If v is unique, then the G-representation generated by v, say S, is simple, for otherwise one could use Lie-Kolchin on a subrepresentation of S, and get a multiple of v as eigenvector. These modules then give all simples, by highest weight theory. This argument does not provide an explicit description of S, which is something one in general does not have, even in the commutative characteristic p case. In Jantzen, this is phrased in cohomological language, using the right derived functors of the induction functor, see Corollary II.2.3 [47]. To apply these ideas in our (non-commutative!) setting, we need some machinery.

Definition 6.3.11 *A k-coalgebra C is called filtered if $C = \cup_{n\geq 0} C_n$, where $(C_n)_n$ is an ascending chain of subspaces, satisfying*

$$\Delta(C_n) \subset \sum_{m\geq 0} C_m \otimes C_{n-m}. \tag{6.5}$$

Lemma 6.3.12 *For a filtered k-coalgebra C, and a C-comodule V, there exists a non-trivial subspace V_0 that is a C_0-comodule.*

Proof: The coaction $\delta : V \to C \otimes V$ of any element $v \in V$ can be decomposed in a way respecting the filtration:

$$\delta(v) = \sum_{n,i} c_{n,i} \otimes v_{n,i},$$

if we take the $(c_{n,i})_{n,i}$ to be preimages of the bases $(\bar{c}_n)_i$ of C_n/C_{n-1} for the natural quotient maps $C_n \to C_n/C_{n-1}$. Now define V_0 to be the span of $(v_{N,i})_i$, with N maximal among the n for which there exists a non-zero $v_{n,i}$ in $\delta(v)$. Since V is a comodule, we have

$$\sum_{n,i} \Delta(c_{n,i}) \otimes v_{n,i} = \sum_{n,i} c_{n,i} \otimes \delta(v_{n,i}) \in C \otimes C \otimes V.$$

Reducing to $C_N/C_{N-1} \otimes C \otimes V$, and noticing that because C is filtered, the comultiplication descends to a map $\bar{\Delta} : C_N/C_{N-1} \to C_N/C_{N-1} \otimes C_0$, the above equality provides us with the inclusion

$$\sum_i \bar{c}_{N,i} \otimes \delta(v_{N,i}) \subset C_N/C_{N-1} \otimes C_0 \otimes V_0.$$

Since the $(\bar{c}_{N,i})_i$ form a basis, we have that $\delta(V_0) \subset C_0 \otimes V_0$. □

Remark that all group like elements g of a filtered k-coalgebra lie in C_0: if $g \in C_N \backslash C_{N-1}$, then $0 \neq \bar{\Delta}(g) = g \otimes g \in C_N/C_{N-1} \otimes C_N/C_{N-1}$. This contradicts (6.5), unless $N = 0$. Given a group like element g in a coalgebra B, and V a B-comodule, a non-zero vector $v \in V$ is called a semi-invariant with weight g if

$$\delta(v) = g \otimes v.$$

If V is actually a subrepresentation of the regular representation B, $\delta = \Delta$, and by applying $1 \otimes \epsilon$ to the previous equation and using counitality on the left hand side, we find that $v = g\epsilon(v)$, and v is a scalar multiple of g.

From now on we will consider $\mathcal{O}_{nc}(GL_2)$. Set $\mathcal{O}_{nc}(B) = \mathcal{O}_{nc}(GL_2)/(b)$; this is still a Hopf algebra since

$$\Delta(b) = a \otimes b + b \otimes d, \;\; S(b) = -\delta^{-1} b.$$

The notation is obvious: $\mathcal{O}_{nc}(B)$ should be viewed as an analogue of the coordinate ring of the Borel subgroup of lower triangular matrices of GL_2. Its defining equations are

$$ca = ac \qquad da = ad = \delta \qquad \delta \cdot \delta^{-1} = \delta^{-1} \cdot \delta = 1$$
$$a\delta^{-1}d = d\delta^{-1}a = 1 \qquad c\delta^{-1}d = d\delta^{-1}c$$

In particular, a and d are invertible, so we can rewrite $\mathcal{O}_{nc}(B)$ as

$$\mathcal{O}_{nc}(B) = k\langle c, d, d^{-1}\rangle[a, a^{-1}].$$

Lemma 6.3.13 *$\mathcal{O}_{nc}(B)$ is a filtered coalgebra. If $\mathcal{O}_{nc}(B)'_n \subset \mathcal{O}_{nc}(B)$ is the span of the monomials in a, c and d that contain n c's, then*

$$\mathcal{O}_{nc}(B)_n = \oplus_{m \leq n} \mathcal{O}_{nc}(B)'_m$$

Proof: Notice that $\mathcal{O}_{nc}(B)'_0$ is spanned by the products $a^s d^t$. We have

$$\Delta(c) = c \otimes a + d \otimes c,$$

so for $x, y \in \mathcal{O}_{nc}(B)'_0$

$$\Delta(xcy) \subset \sum_{x,y} x_{(1)}cy_{(1)} \otimes x_{(2)}ay_{(2)} + x_{(1)}dy_{(1)} \otimes x_{(2)}cy_{(2)}.$$

This means that

$$\Delta(\mathcal{O}_{nc}(B)'_1) \subset \mathcal{O}_{nc}(B)'_1 \otimes \mathcal{O}_{nc}(B)'_0 + \mathcal{O}_{nc}(B)'_0 \otimes \mathcal{O}_{nc}(B)'_1,$$

and since $\mathcal{O}_{nc}(B)'_n = (\mathcal{O}_{nc}(B)'_1)^n$, by induction this gives

$$\Delta(\mathcal{O}_{nc}(B)'_n) \subset \oplus_{m\geq 0}\mathcal{O}_{nc}(B)'_m \otimes \mathcal{O}_{nc}(B)'_{n-m}.$$

This means that $\mathcal{O}_{nc}(B) = \oplus_{n\geq 0}\mathcal{O}_{nc}(B)'_n$ is an $\mathbb{N}$-graded coalgebra, so $\mathcal{O}_{nc}(B)$ becomes filtered by setting

$$\mathcal{O}_{nc}(B)_n = \oplus_{m\leq n}\mathcal{O}_{nc}(B)'_m.$$

□

By the discussion preceding the lemma, we know that all group like elements of $\mathcal{O}_{nc}(B)$ are contained in, and thus equal to $\mathcal{O}_{nc}(B)_0 = k[a, a^{-1}, d, d^{-1}]$, the (commutative!) coordinate ring of a torus. Consider $V = ka + kc \subset \mathcal{O}_{nc}(GL_2)$ as subrepresentation of the regular representation, and also $R = k\delta \subset \mathcal{O}_{nc}(GL_2)$. To any word in a and δ, say

$$w = a^{m_1}\delta^{n_1}a^{m_2}\cdots a^{m_{k-1}}\delta^{n_{k-1}}a^{m_k},\ m_i \in \mathbb{N},\ n_i \in \mathbb{Z} - \{0\},$$

we can associate the $\mathcal{O}_{nc}(GL_2)$ representation

$$W_w = V^{m_1}R^{n_1}\cdots V^{m_k}R^{n_k} \subset \mathcal{O}_{nc}(GL_2).$$

Lemma 6.3.14 *Every $\mathcal{O}_{nc}(B)$ representation contains a semi-invariant.*

Proof: From Lemma 6.3.12 we know that every $\mathcal{O}_{nc}(GL_2)$ comodule V contains a $\mathcal{O}_{nc}(GL_2)_0$ comodule V_0. Since $\mathcal{O}_{nc}(GL_2)_0$ is the commutative coordinate ring of a torus, V_0 is spanned by semi-invariants: the simple torus representations are given by

$$k_{(\alpha,\beta)} \to k[a, a^{-1}, d, d^{-1}] \otimes k_{(\alpha,\beta)} : v \mapsto a^\alpha d^\beta \otimes v,$$

for $(\alpha, \beta) \in \mathbb{Z}^2$. □

Proposition 6.3.15 *Every subrepresentation of W_w contains w.*

Proof: The composition

$$W_w \to \mathcal{O}_{nc}(GL_2) \to \mathcal{O}_{nc}(B)$$

is injective, since the image of a basis vector, say

$$a^{p_1}c^{q_1}\delta^{n_1}a^{p_2}\cdots\delta^{n_{k-1}}a^{p_k}c^{q_k},$$

where $p_i + q_i = m_i$, is given by

$$a^{\sum p_i + \sum n_i}c^{q_1}d^{n_1}c^{q_2}\cdots c^{q_{k-1}}d^{n_{k-1}}c^{q_k}. \tag{6.6}$$

Hence, every basis vector of W_w gets sent to a different basis vector of $\mathcal{O}_{nc}(B)$, the map is injective, and we can view W_w as subrepresentation of $\mathcal{O}_{nc}(B)$. From our discussion on semi-invariants, it follows that W_w contains at most one semi-invariant of a fixed weight $a^u d^l$ (up to scalar multiple), and this is then $a^u d^l$, if this

element sit in W_w. From (6.6), we see immediately that only one weight, and thus only one semi-invariant can and does occur, namely

$$a^{\sum m_i + \sum n_i} d^{\sum n_i}.$$

Looking at this by converting W_w to a submodule of C again, we see that w is the unique D-semi-invariant in W_w. Suppose now that V is a subrepresentation of W_w; by the lemma, we know that V contains a D-semi-invariant v. Then by the previous considerations, v must be a multiple of w, proving the proposition. □

Now we are finally ready to obtain the analogue to the statement we discussed in the introduction.

Proposition 6.3.16 *The subrepresentation S_w of W_w cogenerated by w is simple.*

Proof: If it were not simple, it would contain some subrepresentation which, by the previous proposition, would contain w and thus S_w. □

This approach to the construction of simple $\mathcal{O}_{nc}(GL_2)$ representations should be generalizable to the $n > 2$ case, which is what we hope to accomplish in the near future. Notice that we haven't fully solved the representation theory problem here: we haven't proven that every representation occurs as some subrepresentation of a W_w.

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