ULTRADIFFERENTIABLE CLASSES AND THE DENJOY-CARLEMAN THEOREM

Let I be a closed compact interval in $\mathbb{R}$ and let $x_0 \in I$. A function $f : I \to \mathbb{C}$ is called *real analytic* if there exists a complex neighbourhood Ω of $[-1, 1]$ and a holomorphic function F on Ω such that $F_{|I} = f$. The uniqueness property of holomorphic functions implies that for all real analytic functions f on I the following property holds

$$D^p f(x_0) = 0, \forall p \in \mathbb{N} \Longrightarrow f = 0 \text{ on } I. \tag{0.1}$$

In 1901 E. Borel showed that the uniqueness property (0.1) does not characterise the class of real analytic functions by constructing a class of functions containing functions that are not real analytic but that does satisfy (0.1). He called such classes *quasianalytic*. This raises the question whether quasianalyticity could be characterized in some sense. In order to turn this question into a concrete mathematical problem, we introduce the following spaces.

Definition 1. Let $M = (M_p)_{p \in \mathbb{N}}$ be a sequence of positive numbers. We define $C^{\{M\}}(I)$ as the space of all $f \in C^\infty(I)$ such that

$$\max_{x \in I} |D^p f(x)| \leq C h^p M_p, \qquad \forall p \in \mathbb{N},$$

for some $h > 0$. $C^{\{M\}}(I)$ is called the space of *ultradifferentiable functions of class* $\{M_p\}$.

Note that, by the Cauchy estimates, the space of real analytic functions coincides with $C^{\{p!\}}(I)$. In 1912 Hadamard posed the following problem: *Find necessary and sufficient conditions on M which ensure that $C^{\{M\}}(I)$ is quasianalytic, i.e., that (0.1) holds for all $f \in C^{\{M\}}(I)$.* Based upon earlier work of Denjoy, Carleman in 1922 completely solved Hadamard's problem by showing the following beautiful result (called the *Denjoy-Carleman theorem*).

Theorem 2. *Let $M = (M_p)_{p \in \mathbb{N}}$ be a sequence of positive numbers*[1]*. The space $C^{\{M\}}(I)$ is quasianalytic if and only if*

$$\sum_{p=0}^{\infty} \frac{1}{M_p^{1/p}} = \infty.$$

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The goal of this project is to study spaces of ultradifferentiable functions and to discuss two different proofs of the Denjoy-Carleman theorem: the original complex analytic proof of Carleman [1] and a more elementary one of Bang that only uses real analysis [2].

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References

[1] P. Koosis, *The Logarithmic Integral I*, Cambridge University Press, 1988.
[2] S. Mandelbrojt, *Séries adhérentes, Régularisation des suites. Applications.* Gauthier-Villars, Paris, 1952.

[1] The sequence M should satisfy certain technical assumptions but we do not discuss them here.