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Approach theory with an application to function spaces

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Abstract

Topologists prefer to work in a category like **TOP** (topological spaces and continuous maps), even if this means that they have to abandon an originally metric setup in the category **MET** (metric spaces and non-expansive maps). A first reason why we prefer to look at the underlying topology, is given by the fact that it is the topology which provides us the framework in which most of the basic concepts of analysis are defined, such as, for example, convergence, continuity, and compactness. A second reason why we prefer **TOP** is because this construct is stable under all usual constructions, whereas **MET** is only stable under formation of subspaces and finite products. This transition, however, does not solve all problems. The most fundamental problem which arises is the problem concerning products. It is well-known that metric initial structures do not necessarily accord with topological initial structures: the countable product of metrizable spaces is metrizable, but there is no canonical metric for the product topology, and for uncountable products there simply is no metric at all for the product topology. This is the main motivation for looking at a common supercategory of **TOP** and **MET**, namely **AP** (the topological construct of approach spaces and contractions), in which we are able to solve this problem. Approach spaces thus are a generalization of both metric spaces, based on point-to-set distances instead of point-to-point distances, and topological spaces. There is more, we are able to prove that **TOP** can be embedded as a full concretely reflective and coreflective subconstruct and that **MET** can be embedded as a full concretely coreflective subconstruct of **AP**. This means that both topological and metric spaces can be studied in **AP** as objects on equal footing.

Now that we have such a common supercategory, which solves the problem concerning products, there is much more we can study in this category. A first idea which presents itself is to lift certain topological properties (such as, for example, compactness, connectedness and regularity) and metric concepts (such as, for example, total boundedness and Cantor-connectedness) to the realm of approach theory. A second idea is to lift well-known theorems in topology (such as, for example, extension theorems or the theorem of Ascoli) to the realm of approach theory. We expect that this will give us much stronger and much more widely applicable theorems, which will enable us to deal with much more examples then the categories **TOP** and **MET** have to offer.

Contents

INTRODUCTION

Before diving in to the heart of the matter, we give some motivations why we should study approach spaces and approach theory. Most of these motivations are inspired by the book about approach theory by R. Lowen [12], which serves as a guideline for the first three chapters and the first two sections of the fourth chapter. Apart from those motivations, we will also introduce a new one concerning regularity, a topic which will be studied in this work and which was not studied in [12].

As mathematicians, we are all familiar with the construct **MET** (the construct of metric spaces and non-expansive maps). There is absolutely no doubt that metric spaces play an important role in mathematics. Nevertheless it is topology which provides the framework in which most of the basic concepts of analysis are defined, such as, for example, convergence, continuity and compactness. As topologists, we prefer to work in a construct like **TOP** (construct of topological spaces and continuous maps). The main reason why is because **TOP** is stable under all usual constructions such as subspaces, products, quotients, and coproducts, whereas **MET** is stable only under the formation of subspaces and finite products. One may say that this problem can easily be avoided by going from metrics to extended pseudometrics (metrics where the value infinity is allowed and which do not necessarily have an underlying topology which is Hausdorff). The construct of extended pseudometric spaces and non-expansive maps will be denoted by $p\mathbf{MET}^{\infty}$. This transition, however, does not solve all problems. A fundamental problem which remains is that metric initial structures do not necessarily accord with topological initial structures.

We will now take a closer look at this problem concerning products. Consider therefore the construct $pq\mathbf{MET}^{\infty}$ of extended pseudo-quasimetric spaces (a quasimetric is a metric which does not have to be symmetric) and non-expansive maps. As we know there is a natural functor

$$pq\mathbf{MET}^{\infty} \to \mathbf{TOP}$$

which takes an ∞pq-metric space to its underlying topological space and leaves morphisms unaltered. This underlying topology functor commutes with the formation of finite products. If we take a finite family of metric spaces and make their product in $pq\mathbf{MET}^{\infty}$, then the underlying topology is the product topology. The underlying topology functor, however, does not commute with infinite products. There are two cases to be considered: countable and uncountable products. If we

consider a countable family of ∞pq-metric spaces $(X_n, d_n)_{n\in\mathbb{N}}$ then the product topology on $\prod_{n\in\mathbb{N}} X_n$ can be metrized by a host of ad hoc ∞pq-metrics of various types, such as, for example

$$d(x,y) := \sum_{n\in\mathbb{N}} a_n\big(d_n(x_n,y_n) \wedge b_n\big),$$

or

$$d(x,y) := \sup\big\{a_n\big(d_n(x_n,y_n) \wedge b_n\big) \mid n \in \mathbb{N}\big\},$$

for suitable choices of the sequences $(a_n)_n$ and $(b_n)_n$, but $\prod_{n\in\mathbb{N}} X_n$ is never metrized by the product ∞pq-metric in $pq\mathbf{MET}^\infty$. When we consider an uncountable family of metric spaces, the product space is not metrizable at all, neither by the product ∞pq-metric in $pq\mathbf{MET}^\infty$, nor by any other ∞pq-metric.

We may now wonder why we can produce a canonical ∞pq-metric for finite products, only ad hoc ∞pq-metrics for countable products, and no ∞pq-metrics for uncountable products. We want to find out whether it is really the case that, unless we have a finite number of factors, all canonical notion of distance in the product space has vanished. We also want to find out whether it is possible to find a suitable construct which has both $pq\mathbf{MET}^\infty$ and $\mathbf{TOP}$ as subconstructs and which solves this problem concerning products. In other words, if we consider the following diagram

$$\begin{tikzcd} & ? & \\ pq\mathbf{MET}^\infty \arrow[ur, "E_1"] & & \mathbf{TOP} \arrow[ul, "E_2"'] \end{tikzcd}$$

we want to know if we can place a suitable topological construct on the question mark such that both functors E_1 and E_2 become embedding functors and such that the following diagram commutes

$$\begin{tikzcd} & ? \arrow[dr, "R_2"] & \\ pq\mathbf{MET}^\infty \arrow[ur, "E_1"] \arrow[rr, "F"] & & \mathbf{TOP} \end{tikzcd}$$

where E_1 is the same embedding functor as in the first diagram, F is the underlying topology functor, and where R_2 is a functor which we will interpret as the $\mathbf{TOP}$-coreflector related to the embedding E_2

A second motivation comes from numerical analysis and approximation theory. Consider therefore the real line $\mathbb{R}$ with the usual Euclidean topology and consider the sequences $(x_n)_n$ and $(y_n)_n$ defined by

$$x_n := \begin{cases} \epsilon & n \text{ even}, \\ -\epsilon & n \text{ odd}, \end{cases} \quad \text{and} \quad y_n := \begin{cases} n & n \text{ even}, \\ -n & n \text{ odd}. \end{cases}$$

Neither of these sequences converges. The first one, however, has two main convergent subsequences. From the point of view of numerical analysis or approximation theory, the sequence might be considered "approximately convergent" to 0

for sufficiently small ϵ. The second sequence on the other hand has no convergent subsequences, and could not even remotely be considered to be "approximately convergent" to any point in $\mathbb{R}$.

Another, and more striking, example is obtained as follows. Let $\varphi : \mathbb{R} \to]-\epsilon, \epsilon[$ be a homeomorphism, and let $(r_n)_{n\in\mathbb{N}}$ be an enumeration of the rationals. The sequence $(r_n)_n$ is not remotely "approximately convergent" to any point of $\mathbb{R}$. The sequence $(\varphi(r_n))_n$ on the other hand, might again be considered "approximately convergent" for sufficiently small ϵ.

By means of the topology of $\mathbb{R}$, we are unable to detect the different behaviour of these sequences. In order to see the difference we require a metric. In literature on approximation theory concepts have been introduced to deal with these ideas in the setting of metric spaces. There is, however, no canonical theory to classify these sequences according to the above criteria, in the way topological convergence classifies sequences in convergent and nonconvergent sequences. More fundamentally, once we leave the realm of metric spaces we have no mathematical model, canonically or ad hoc, to cope with these ideas.

We now wonder if it is possible to build a theory in which the above ideas in the realm of approximation theory arise as naturally as the theory of convergence arises in the setting of topology. In section 3.2 we will recall the examples which we introduced above and we will then study them more extensively.

A third motivation comes from comparing topological properties to metric concepts. Certain topological properties have remarkable metric counterparts. Compactness, for example, is a topological property which in the case of metrizable topological spaces is closely linked to the metric concept of total boundedness. Moreover, both properties have very similar characterizations. Compactness of a topological space can be characterized by the fact that every open cover should have a finite subcover. Total boundedness, on the other hand, can be characterized in a similar way. A metric space is totally bounded if the collection of all balls with an arbitrary fixed radius has a finite subcollection which still covers the whole space.

A similar situation occurs when looking at the topological property of connectedness. Topological connectedness of a space X can be characterized by the fact that X cannot be split into two nontrivial closed parts. The metric concept of Cantor-connectedness has a similar characterization. A metric space is Cantor-connected if it cannot be split into two nontrivial parts which lie at a strictly positive distance from each other.

We now want to find out whether there exists a unifying theory behind these different but similar concepts.

In contrast to the two foregoing topological properties which do have metric counterparts, we can also look at topological properties which do not seem to have any metric counterpart at all. Take for example the topological property regularity. As introduced in Bourbaki [3], regularity is of purely topological nature and no similar concepts in metric spaces seem to exist. However, if we are able to find a common supercategory of **TOP** and $pq\mathbf{MET}^{\infty}$, as mentioned in the first motivation, we should question whether it is really the case that no metric coun-

terparts can be found. Once we have found an appropriate supercategory, we can lift the concept of regularity to this common supercategory. By lifting regularity to this level, we will be able to study this property in the metric context, and an appropriate metric counterpart will reveal itself.

A fourth and last motivation is inspired by Kuratowski's measure of noncompactness. The Kuratowski measure of noncompactness for a p-metric space (X, d) is defined by

$$m_K(A) := \inf \Big\{ \epsilon \in \mathbb{R}^+ \mid \exists X_1, \cdots, X_n \subset X : \max_{i=1}^{n} \operatorname{diam}(X_i) \leq \epsilon, A \subset \bigcup_{i=1}^{n} X_i \Big\}.$$

Since this measure was first introduced by Kuratowski, several variants have been introduced in the literature, including the Hausdorff or ball measure of noncompactness, defined by

$$m_H(A) := \inf \Big\{ \epsilon \in \mathbb{R}^+ \mid \exists x_1, \cdots, x_n \in X : A \subset \bigcup_{i=1}^{n} B(x_i, \epsilon) \Big\}.$$

For all these measures the deviation a subset has from being compact is actually more the deviation it has from being totally bounded. So here too one sees certain blending of topological an metric concepts.

We want to find out whether there is a canonical theory where quantified topological concepts exist in the same way as topological concepts exist in **TOP**. Eventually this will lead to quantified theorems. As an example, we will focus on the Ascoli theorem.

In this work, namely in the first four chapters, we will show that all these points mentioned in the items above can be addressed in a satisfactory way and that all problems mentioned can be solved within the setting of what we call approach theory.

In the first chapter we introduce four different structures by which we can define the construct **AP** of approach spaces. These structures will be conceptually very different, but nevertheless equivalent. We will prove all necessary transitions from one structure to another. All transition formulas will be displayed in a table at the end of section 1.5. Further, in this chapter, we also define the morphisms in **AP** and show that **AP** is a nice construct. We show that it is a topological construct in the sense of Adámek *et al.* [1].

In the second chapter we prove that **TOP** can be seen as a nice subconstruct of **AP**. We will show that it can be embedded as a simultaneously concretely reflective and concretely coreflective subconstruct, which is a remarkable result since **TOP** itself does not have any subconstructs which are simultaneously reflectively and coreflectively embedded in **TOP**. The coreflection will be the most important one, since it will allow us to interpret the **TOP**-coreflection of an approach space in the same way that we interpret the topology underlying a metric.

In the third chapter we prove that $pq\mathbf{MET}^\infty$ can also be viewed as a nice subconstruct of **AP**. We will show that it can be embedded as a concretely coreflective subconstruct. Here $pq\mathbf{MET}^\infty$ will not be embedded as a concretely reflective subconstruct, more precisely $pq\mathbf{MET}^\infty$ is not stable under the formation of arbitrary initial structures. This will turn out to be the right of existence of approach spaces. In this chapter we will also give special attention to the motivation mentioned in the second item concerning the existence of a canonical theory of approximate convergence. In particular, we will retake the examples mentioned in the motivation.

In the fourth chapter we will first of all address the motivations raised in the third and fourth item. We will show that compactness and total boundedness are special instances of a unifying property in **AP** and show that connectedness and Cantor-connectedness too are special instances of a unifying property in **AP**. Moreover, we show that the Hausdorff measure of noncompactness generalizes the canonical measure of compactness in **AP**. Following the same philosophy we construct a measure of connectedness in **AP** which has equally good properties as the measure of compactness.

In the last section of this chapter we also introduce a notion of regularity in approach spaces. To this end we focus on the doctoral dissertation of K. Robeys [14], the paper on convergence approach spaces by P. Brock and D. Kent [5], a preprint of the book about index theory by R. Lowen [13] and the paper on regularity in approach theory by B. Banaschewski, R. Lowen and C. Van Olmen [2]. We will give various characterizations of regularity in approach spaces for all of the different types of structures which we will introduce in the first chapter and describe the relation between regularity in approach spaces and regularity in topological spaces. We will also be able to find a metric counterpart of regularity. This will turn out to be the symmetry of the metric.

We will end this section about regularity with an interesting application. We will generalize a well-known extension theorem, as stated in Bourbaki [3], to the realm of approach theory. This theorem was first proved by Jäger in a forthcoming paper about extensions of contractions and uniform contractions on dense subspaces [8].

In the fifth and last chapter of this work, we present an application of approach theory to function spaces. In this chapter we will lift the Ascoli theorem to the realm of approach theory. The theorem of Ascoli describes (pre)compact subsets of function spaces. The classical theorem of Ascoli, as we can find in Bourbaki [4], states the following. Let X, Y be uniform spaces, let Σ be a cover of X which is closed under finite unions and let $\mathcal{H}$ be any collection of functions from X to Y. If each set in Σ is precompact, if for each $A \in \Sigma$ the collection of $\mathcal{H}_{|A}$ is uniformly equicontinuous and if for each $x \in X$, $\mathrm{ev}_x(\mathcal{H})$ is precompact, then $\mathcal{H}$ is precompact.

It is clear that, before we can prove this theorem, we need to extend the setting of uniform spaces to approach theory. Therefore, we start chapter 5 with a study of uniform gauge spaces. We will also need a quantified version of precompactness

in approach theory. To this end, we will construct the measure of precompactness. We will compare this definition with the definition of precompactness in uniform spaces and give some good consistency results. Finally, we will need a notion of equicontractivity. We will define when a set $\mathcal{H} \subset Y^X$ is uniformly equicontractive and link this to the uniform concept of uniform equicontinuity. We will also introduce a measure of uniform contractivity and uniform equicontractivity.

This all will enable us to prove a general Ascoli theorem in approach theory. We will see that this theorem, just as many other theorems on the approach level, appears to have no conditions, meaning that the conditions are encapsulated in the inequality formulating the theorem.

For this chapter we refer to the paper 'An Ascoli theorem in approach theory' by R. Lowen [10].

Chapter 1

APPROACH SPACES

In the first chapter we will introduce the structures which will determine what we will call an approach space. Approach spaces can be defined by conceptually very different but nevertheless equivalent structures. Here we will introduce four such structures: distances, limit operators, gauges and towers. Note that these are not the only structures that can determine an approach space, but these will appear to be the most useful for the rest of this work.

After introducing all different structures, we will define the category **AP** and show that this is a topological construct.

For this chapter, we refer to the first chapter in the book on approach spaces by R. Lowen [12].

1.1 Distances

The first and probably most appealing structure which we will be considering is that of a distance between points and sets. In a metric space (X, d) a distance between pairs of points is given and a distance between points and sets can be derived using the following formula

$$\delta_d(x, A) := \inf_{a \in A} d(x, a) \quad \forall x \in X, \forall A \in 2^X.$$

Here we will start from a concept of distance between points and sets. Before giving the precise definition we need to introduce the following notation.

If X is a set, and we have a function $\delta : X \times 2^X \to [0, \infty]$, then for any subset $A \subset X$ and any $\epsilon \in [0, \infty]$, we define

$$A^{(\epsilon)} := \{x \in X \mid \delta(x, A) \leq \epsilon\}.$$

Definition 1.1.1 *A function*

$$\delta : X \times 2^X \to [0, \infty]$$

is called a ***distance*** *on X if it satisfies the following properties:*

(D1) $\forall x \in X : \delta(x, \{x\}) = 0,$

(D2) $\forall x \in X : \delta(x, \emptyset) = \infty$,

(D3) $\forall x \in X, \forall A, B \in 2^X : \delta(x, A \cup B) = \min(\delta(x, A), \delta(x, B))$,

(D4) $\forall x \in X, \forall A \in 2^X, \forall \epsilon \in [0, \infty] : \delta(x, A) \leq \delta\big(x, A^{(\epsilon)}\big) + \epsilon$.

In the same way as in metric spaces the value $\delta(x, A)$ is interpreted as the distance from the point x to the set A. This distance, however, is not necessarily derivable from the distances $\delta(x, \{a\})$, for $a \in A$.

The codomain of a distance, and also many other functions which we will consider, is $[0, \infty]$. It should be pointed out that we will use the symbols $+$ and $-$ for the natural extensions of these operations. More precisely, $+$ and $-$ stand for the usual addition and subtraction in the case of real numbers, and further, for any $x \in [0, \infty[$ we have $x + \infty = \infty + x = \infty + \infty = \infty$, $\infty - x = \infty$ and $\infty - \infty = 0$.

The following proposition contains some fundamental properties which we will use in the sequel.

Proposition 1.1.2 *If $\delta : X \times 2^X \to [0, \infty]$ is a distance on X then the following hold:*

1. $\forall x \in X, \forall A \in 2^X : x \in A \Rightarrow \delta(x, A) = 0$,
2. $\forall x \in X, \forall A, B \in 2^X : A \subset B \Rightarrow \delta(x, B) \leq \delta(x, A)$,
3. $\forall x \in X, \forall \mathcal{A} \subset 2^X, \mathcal{A}$ *finite:* $\delta(x, \bigcup \mathcal{A}) = \min_{A \in \mathcal{A}} \delta(x, A)$,
4. $\forall x \in X, \forall A, B \in 2^X : \delta(x, A) \leq \delta(x, B) + \sup_{b \in B} \delta(b, A)$.

Proof.

1. Consider $x \in X$ and choose $A \in 2^X$ such that $x \in A$. The result follows from the following equalities.

$$\begin{aligned} \delta(x, A) &= \delta(x, \{x\} \cup A) \\ &= \min(\delta(x, \{x\}), \delta(x, A)) \\ &= \min(0, \delta(x, A)) \\ &= 0. \end{aligned}$$

2. Let $x \in X$ and choose $A, B \in 2^X$ such that $A \subset B$. Then the following equalities hold:

$$\begin{aligned} \delta(x, B) &= \delta(x, B \cup A) \\ &= \min(\delta(x, B), \delta(x, A)) \\ &\leq \delta(x, A). \end{aligned}$$

3. Let $x \in X, \mathcal{A} \subset 2^X, \mathcal{A}$ finite. $\mathcal{A}$ is finite, so there exist $A_1, A_2, \ldots, A_n \in 2^X$ such that $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$. Then we have

$$
\begin{aligned}
\delta\Big(x, \bigcup \mathcal{A}\Big) &= \delta(x, A_1 \cup A_2 \cup \ldots \cup A_n) \\
&= \min\big(\delta(x, A_1), \delta(x, A_2 \cup \ldots \cup A_n)\big) \\
&= \min\big(\delta(x, A_1), \min(\delta(x, A_2), \delta(x, A_3 \cup \ldots \cup A_n))\big) \\
&= \min\big(\delta(x, A_1), \delta(x, A_2), \delta(x, A_3 \cup \ldots \cup A_n)\big) \\
&= \ldots \\
&= \min\big(\delta(x, A_1), \ldots, \delta(x, A_n)\big) \\
&= \min_{A \in \mathcal{A}} \delta(x, A),
\end{aligned}
$$

which proves the third proposition.

4. Let $x \in X$ and choose $A, B \in 2^X$. Let

$$\epsilon := \inf\big\{\theta \in [0, \infty] \mid B \subset A^{(\theta)}\big\}.$$

Then it follows that

$$
\begin{aligned}
\delta(x, A) &\leq \delta\big(x, A^{(\epsilon)}\big) + \epsilon \\
&\leq \delta(x, B) + \epsilon \\
&\leq \delta(x, B) + \sup_{b \in B} \delta(b, A).
\end{aligned}
$$

□

Although a distance is defined as a function in two variables, points and subsets, it will sometimes be useful to consider the following associated function on points.

For a given subset $A \subset X$ we define

$$\delta_A : X \to [0, \infty] : x \mapsto \delta(x, A).$$

We will call such functions **distance functionals**.

1.2 Limit operators

The next structure provides us with a notion of convergence. The basic set-theoretical tools needed to define convergence are sequences, nets and filters. Sequences suffice in metric spaces, but it will soon be clear that they do not suffice in our case. Although in particular cases we will sometimes give results for sequences, for the general theory we have to turn to either filters or nets. We have chosen to use filters, but all results can be rephrased in terms of nets.

Whenever convenient we will use the following notations. $\mathbf{F}(X)$ will stand for the set of all filters on X, and $\mathbf{U}(X)$ will stand for the set of all ultrafilters on X. Sometimes it will be useful to generalize this notation in the following way. If $\mathcal{F}$ is a given filter on X, then we will denote by $\mathbf{F}(\mathcal{F})$ the collection of all filters on X which are finer than $\mathcal{F}$, and by $\mathbf{U}(\mathcal{F})$ the collection of all ultrafilters on X

which are finer than $\mathcal{F}$. If $\mathcal{F}$ is the trivial filter on X, i.e. $\mathcal{F} = \{X\}$, then $\mathbf{F}(\mathcal{F})$ reduces to $\mathbf{F}(X)$ and $\mathbf{U}(\mathcal{F})$ reduces to $\mathbf{U}(X)$.

If $\mathcal{A}$ is a collection of subsets of X, then the stack of $\mathcal{A}$ is defined as

$$\operatorname{stack} \mathcal{A} := \{B \subset X \mid \exists A \in \mathcal{A} : A \subset B\}.$$

If $\mathcal{F}$ is a filter on X, then the sec of $\mathcal{F}$ is defined as

$$\sec \mathcal{F} := \bigcup_{\mathcal{U} \in \mathbf{U}(\mathcal{F})} \mathcal{U} = \{A \subset X \mid \forall F \in \mathcal{F} : A \cap F \neq \emptyset\}.$$

If $\mathcal{A}$ is a filterbasis, then $\operatorname{stack} \mathcal{A}$ is the filter generated by $\mathcal{A}$. In case $\mathcal{A}$ reduces to a single set A we write $\operatorname{stack} A$ instead of $\operatorname{stack}\{A\}$, and in case the single set A furthermore reduces to a single point a we write $\operatorname{stack} a$ instead of $\operatorname{stack}\{a\}$. If $A \in 2^X$, $A \neq \emptyset$, then $\operatorname{stack} A$ is the filter consisting of all sets containing A, and then, for simplicity, we will often write $\mathbf{F}(A)$ instead of $\mathbf{F}(\operatorname{stack} A)$, and $\mathbf{U}(A)$ instead of $\mathbf{U}(\operatorname{stack} A)$.

Consider a filter $\mathcal{F} \in \mathbf{F}(X)$. For a function $f : X \to Y$, we define the image of a filter as the filter on Y defined by $\operatorname{stack}\{f(F) \mid F \in \mathcal{F}\}$ and we will denote this filter by $f(\mathcal{F})$. This definition makes sure that we do not always have to write stack when considering images of filters, which will make our formulas easier to work with.

The axioms for a limit operator which we are about to introduce are similar to the axioms for a topological convergence structure. One of the axioms requires the so-called **Kowalsky diagonal operation**. Given a filter ψ on $\mathbf{F}(X)$, i.e. $\psi \in \mathbf{F}^2(X)$, we define $\Sigma\psi$ as follows

$$\Sigma\psi = \bigcup_{\mathcal{A} \in \psi} \bigcap_{\mathcal{W} \in \mathcal{A}} \mathcal{W} = \{A \subset X \mid \tilde{A} \in \psi\}$$

with $\tilde{A} = \{\mathcal{W} \in \mathbf{F}(X) \mid A \in \mathcal{W}\}$. We can even extend this definition to ultrafilters Θ on $\mathbf{U}(X)$, i.e. $\Theta \in \mathbf{U}^2(X)$. Here $\Sigma\Theta$ is defined in exactly the same way:

$$\Sigma\Theta = \bigcup_{\mathcal{A} \in \Theta} \bigcap_{\mathcal{W} \in \mathcal{A}} \mathcal{W} = \{A \subset X \mid \tilde{A} \in \Theta\},$$

with $\tilde{A} = \{\mathcal{W} \in \mathbf{U}(X) \mid A \in \mathcal{W}\}$. Later on, in proposition 1.2.3, we will show that $\Sigma\Theta$ is also an ultrafilter.

Consider now a non-empty set J and a function $\sigma : J \to \mathbf{F}(X)$. This function gives us a family of filters $(\sigma(j))_{j \in J}$ on X. If we now consider a filter $\mathcal{F}$ on J, then we are able to apply the Kowalsky diagonal operation Σ to $\sigma\mathcal{F}$. This gives us

$$\begin{aligned}
\Sigma\sigma\mathcal{F} &= \{A \subset X \mid \tilde{A} \in \sigma\mathcal{F}\} \\
&= \bigcup_{\mathcal{A} \in \sigma\mathcal{F}} \bigcap_{\mathcal{W} \in \mathcal{A}} \mathcal{W} \\
&= \bigcup_{F \in \mathcal{F}} \bigcap_{\mathcal{W} \in \sigma F} \mathcal{W} \\
&= \bigcup_{F \in \mathcal{F}} \bigcap_{j \in F} \sigma(j).
\end{aligned}$$

If we take J equal to X, $\sigma : X \to \mathbf{F}(X)$ gives us a family of filters $(\sigma(x))_{x\in X}$ on X indexed by the points of X. In this case the function σ is called a **selection of filters on** X. If we wish to emphasize the fact that we work with a selection of filters on X, we will often denote this selection in a functional way by

$$\mathcal{S} : X \to \mathbf{F}(X) : x \mapsto \mathcal{S}(x),$$

where we denote the function which defines the selection by $\mathcal{S}$ instead of σ.

Definition 1.2.1 *A function*

$$\lambda : \boldsymbol{F}(X) \to [0, \infty]^X$$

is called a ***limit operator*** *on X if it satisfies the following properties:*

(L1) $\forall x \in X : \lambda(\operatorname{stack} x)(x) = 0$,

(L2) $\forall \mathcal{F}, \mathcal{G} \in \boldsymbol{F}(X) : \mathcal{F} \subset \mathcal{G} \Rightarrow \lambda\mathcal{G} \leq \lambda\mathcal{F}$,

(L3) for any family $(\mathcal{F}_j)_{j\in J}$ of filters on X:

$$\lambda\Big(\bigcap_{j\in J} \mathcal{F}_j\Big) = \sup_{j\in J} \lambda\mathcal{F}_j,$$

(L4) for any $\mathcal{F} \in \boldsymbol{F}(X)$ and any selection of filters $\mathcal{S} : X \to \boldsymbol{F}(X)$ on X:

$$\lambda\Sigma\mathcal{S}\mathcal{F} \leq \lambda\mathcal{F} + \sup_{x\in X} \lambda(\mathcal{S}(x))(x).$$

For any $\mathcal{F} \in \boldsymbol{F}(X)$, the function

$$\lambda\mathcal{F} : X \to [0, \infty]$$

is called the ***limit*** *of $\mathcal{F}$.*

The value $\lambda\mathcal{F}(x)$ is interpreted as the distance that the point x is away from being a limit point of the filter $\mathcal{F}$. The word distance here is meant intuitively and not in the sense of definition 1.1.1.

The smaller the value $\lambda\mathcal{F}(x)$, the closer x becomes to being a limit point of x. Of course, at this point this is merely an intuitive and as yet unsupported interpretation of an abstract concept, but both theory and examples will justify the use of this interpretation. Notice also that (L2) is actually a consequence of (L3).

Before moving on to the next structure we give some preliminary properties related to filters. The following is a useful, purely filter-theoretic result which we will require on several occasions.

Proposition 1.2.2 *If $\mathcal{F} \in \boldsymbol{F}(X)$, and for each ultrafilter $\mathcal{U} \in \boldsymbol{U}(\mathcal{F})$, $\sigma(\mathcal{U}) \in \mathcal{U}$ then there exists a finite set $U_\sigma \in \boldsymbol{U}(\mathcal{F})$ such that*

$$\bigcup_{\mathcal{U}\in U_\sigma} \sigma(\mathcal{U}) \in \mathcal{F}.$$

Proof. Suppose the conclusion does not hold. Then this implies that the family

$$\mathcal{F} \cup \{X \setminus \sigma(\mathcal{U}) \mid \mathcal{U} \in \mathbf{U}(\mathcal{F})\}$$

has the finite intersection property, and thus is contained in some ultrafilter $\mathcal{U} \in \mathbf{U}(\mathcal{F})$. This, however, implies that both $\sigma(\mathcal{U}) \in \mathcal{U}$ and $X \setminus \sigma(\mathcal{U}) \in \mathcal{U}$, which is a contradiction. □

We will also need the following results concerning the diagonal operation.

Proposition 1.2.3 *Let J be a non-empty set and consider a function $\sigma : J \to \boldsymbol{F}(X)$ and a filter $\mathcal{F} \in \boldsymbol{F}(J)$. Then the following properties hold:*

1. *If $(\mathcal{G}_l)_{l \in L}$ is a family of filters on J, and $\mathcal{F} = \bigcap_{l \in L} \mathcal{G}_l$, then*

$$\Sigma\sigma\mathcal{F} = \bigcap_{l \in L} \Sigma\sigma\mathcal{G}_l.$$

2. $\Sigma\sigma\mathcal{F} = \bigcap_{(\mathcal{U}_j)_{j \in J} \in \prod_{j \in J} \boldsymbol{U}(\sigma(j))} \bigcup_{F \in \mathcal{F}} \bigcap_{j \in F} \mathcal{U}_j$.

3. *If all filters involved are ultrafilters, this means that $\sigma : J \to \boldsymbol{U}(X)$ and $\mathcal{F} \in \boldsymbol{U}(J)$, then so is $\Sigma\sigma\mathcal{F}$.*

Proof. The proof of the first property is purely set-theoretical.

We first prove the second property. One inclusion is clear. To show the other one, suppose that $A \notin \Sigma\sigma\mathcal{F}$. For any $j \in J$, we now choose an ultrafilter in the following way:

$$\begin{cases} \text{if } A \notin \sigma(j) & \text{choose } \mathcal{U}_j \in \mathbf{U}(\sigma(j)) \text{ such that } A \notin \mathcal{U}_j, \\ \text{if } A \in \sigma(j) & \text{choose } \mathcal{U}_j \in \mathbf{U}(\sigma(j)) \text{ arbitrary.} \end{cases}$$

Then since $A \notin \Sigma\sigma\mathcal{F}$, it follows that, for any $F \in \mathcal{F}$, there exists $j \in F$ such that $A \notin \sigma(j)$, and, hence, $A \notin \mathcal{U}_j$. Consequently, $A \notin \bigcup_{F \in \mathcal{F}} \bigcap_{j \in F} \mathcal{U}_j$.

To prove the last property we use the following characterization of ultrafilters: $\mathcal{U}$ is an ultrafilter on X iff $\mathcal{U}$ is a filter on X and for every subset $A \subset X$ we have $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$. Suppose that all filters involved are ultrafilters and that $A \subset X$. We have to prove that

$$A \in \Sigma\sigma\mathcal{F} \text{ or } X \setminus A \in \Sigma\sigma\mathcal{F}.$$

Suppose $A \notin \Sigma\sigma\mathcal{F}$. This means that for all $F \in \mathcal{F}$ there exists $j \in J$ such that $A \notin \sigma(j)$. Consider the set $B := \{j \in J \mid X \setminus A \in \sigma(j)\}$. We first prove that $B \in \mathcal{F}$. We have

$$J \setminus B = J \setminus \{j \in J \mid X \setminus A \in \sigma(j)\} = \{j \in J \mid X \setminus A \notin \sigma(j)\} = \{j \in J \mid A \in \sigma(j)\}.$$

If $J \setminus B \in \mathcal{F}$, then we have found $F \in \mathcal{F}$, such that for all $j \in F$ $A \in \sigma(j)$ which would imply that $A \in \Sigma\sigma\mathcal{F}$. Since this is impossible and because of the fact that $\mathcal{F}$ is an ultrafilter, we get $B \in \mathcal{F}$. This means that we have found a set $B \in \mathcal{F}$ such that for all $j \in B$ $X \setminus A \in \sigma(j)$. This implies that $X \setminus A \in \bigcup_{F \in \mathcal{F}} \bigcap_{j \in F} \sigma(j)$, or, in other words, $X \setminus A \in \Sigma\sigma\mathcal{F}$. □

1.3 Gauges

Before moving on to the next structure, we will introduce some terminology concerning metric spaces. Given a set X, a function

$$d : X \times X \to [0, \infty]$$

is called a **metric** if it fulfills the following properties for all $x, y, z \in X$:

1. $d(x, x) = 0$,
2. $d(x, y) \leq d(x, z) + d(z, y)$,
3. $d(x, y) = d(y, x)$,
4. $d(x, y) = 0 \Rightarrow x = y$,
5. $d(x, y) < \infty$.

The second property is referred to as the **triangle inequality**, the third one as **symmetry**, the fourth one as **separatedness**, and the last one as **finiteness**. If d is not necessarily finite, then we call it an **extended metric**, denoted by ∞-metric for short. If it is not necessarily separated, then we call it a **pseudometric**, denoted by p-metric for short, and if it is not necessarily symmetric, then we call it a **quasimetric**, denoted by q-metric for short. Any combination of these is also possible; thus the most general type we consider is an ∞pq-metric, which is a function only fulfilling the first and second properties. A pair (X, d) where d is an ∞pq-metric on X is called an **extended pseudo-quasimetric space**, ∞pq-metric space for short.

We denote by $pq\mathrm{M}^\infty(X)$ (respectively $p\mathrm{M}^\infty(X)$) the set of all ∞pq-metrics (respectively all ∞p-metrics) on X. Both these sets, equipped with the pointwise order, are complete lattices, $p\mathrm{M}^\infty(X)$ actually being a closed sublattice of $pq\mathrm{M}^\infty(X)$.

A lattice-theoretical **ideal** in $pq\mathrm{M}^\infty(X)$ (respectively $p\mathrm{M}^\infty(X)$) is a subset $\mathcal{D}$ which is closed under the formation of taking finite suprema and which is such that if $d \in \mathcal{D}$ and $d' \leq d$, then $d' \in \mathcal{D}$. We call such an ideal an **ideal of ∞pq-metrics** (respectively **ideal of ∞p-metrics**) or an ideal in $pq\mathrm{M}^\infty(X)$ (respectively $p\mathrm{M}^\infty(X)$).

The type of structure which we will introduce now should be compared with the definition of a uniform space via a family of p-metrics, called a uniform gauge.

Given a collection $\mathcal{D} \subset pq\mathrm{M}^\infty(X)$ and a ∞pq-metric $d \in pq\mathrm{M}^\infty(X)$, we will say that d **is dominated by** $\mathcal{D}$ or that $\mathcal{D}$ **dominates** d, if

$$\forall x \in X, \forall \epsilon > 0, \forall \omega < \infty : \exists d_x^{\epsilon,\omega} \in \mathcal{D} \text{ such that } d(x, \cdot) \wedge \omega \leq d_x^{\epsilon,\omega}(x, \cdot) + \epsilon.$$

We will then also say that the family $(d_x^{\epsilon,\omega})_{x \in X, \epsilon > 0, \omega < \infty}$ dominates d.
Further we will say that a collection of ∞pq-metrics $\mathcal{D}$ is **saturated**, if any ∞pq-metric d, which is dominated by $\mathcal{D}$ already belongs to $\mathcal{D}$.

Definition 1.3.1 *A subset $\mathcal{G}$ of $pqM^\infty(X)$ is called a* ***gauge*** *if it is an ideal in $pqM^\infty(X)$ which fulfills the following property:*

(G1) $\mathcal{G}$ is saturated.

It regularly happens that one has a collection of ∞pq-metrics which would be a natural candidate to form a gauge but not all conditions are fulfilled. The following type of collection will often be encountered. We recall that a subset $\mathcal{H}$ of $pq\mathrm{M}^\infty(X)$ is an ideal basis in $pq\mathrm{M}^\infty(X)$ if for any $d, e \in \mathcal{H}$ there exists a $c \in \mathcal{H}$ such that $d \vee e \leq c$.

Definition 1.3.2 *A subset $\mathcal{H}$ of $pqM^\infty(X)$ is called a* ***gauge basis*** *if it is an ideal basis in $pqM^\infty(X)$.*

By definition a gauge is also a gauge basis, and any result shown to hold for gauge bases will also hold for gauges.

In order to derive the gauge from a gauge basis we will require a **saturation operation**. Given a subset $\mathcal{D} \subset pq\mathrm{M}^\infty(X)$, we define

$$\tilde{\mathcal{D}} := \{d \in pq\mathrm{M}^\infty(X) \mid \mathcal{D} \text{ dominates } d\}.$$

We call $\tilde{\mathcal{D}}$ the **saturation** of $\mathcal{D}$.

Definition 1.3.3 *An ideal basis $\mathcal{H}$ in $pqM^\infty(X)$ is said to be a* ***basis for a gauge*** *$\mathcal{G}$ if $\tilde{\mathcal{H}} = \mathcal{G}$. In this case we also say that $\mathcal{H}$ generates $\mathcal{G}$ or that $\mathcal{G}$ is generated by $\mathcal{H}$.*

It is possible for a gauge to be generated by different gauge bases.

Example 1.3.4

Consider the real line $\mathbb{R}$ with its usual metric d_E. Then for any collection of real numbers $A \subset [0, 1]$ such that $\sup A = 1$,

$$\mathcal{H}' := \{ad_E \mid a \in A\}$$

is a gauge basis which generates the same gauge as $\mathcal{H} := \{d_E\}$.

Proof. It is clear that $\mathcal{H}$ and $\mathcal{H}'$ are both ideal bases. Now we prove that $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}'$. First we prove that $\tilde{\mathcal{H}}' \subset \tilde{\mathcal{H}}$. Consider $d \in \tilde{\mathcal{H}}'$. Then for all $x \in X$, for all $\epsilon > 0$ and for all $\omega < \infty$, there exists $a \in A$ such that

$$d(x, \cdot) \wedge \omega \leq ad_E(x, \cdot) + \epsilon.$$

But then we have $\forall x \in X, \forall \epsilon > 0, \forall \omega < \infty$

$$\begin{aligned} d(x, \cdot) \wedge \omega &\leq ad_E(x, \cdot) + \epsilon \\ &\leq \sup A d_E(x, \cdot) + \epsilon \\ &= d_E(x, \cdot) + \epsilon \end{aligned}$$

This means that d is dominated by $\mathcal{H}$ and we get that $d \in \tilde{\mathcal{H}}$.

To prove the other inclusion, consider d_E. Let $x \in X$,$\epsilon > 0$ and $\omega < \infty$. Then we have that

$$d_E(x,\cdot) \wedge \omega \leq d_E(x,\cdot) + \epsilon = \sup A d_E(x,\cdot) + \epsilon.$$

Choose $a \in A$, such that $a \leq \frac{\omega - \epsilon}{\omega}$. We can find such $a \in A$, because of the fact that $\sup A = 1$. For such a, we have that

$$d_E(x,\cdot) \wedge \omega \leq a d_E(x,\cdot) + \epsilon.$$

Hence, we have that $d_E \in \tilde{\mathcal{H}}'$. □

Proposition 1.3.5 *If $\mathcal{H}$ is a gauge basis, then $\tilde{\mathcal{H}}$ is a gauge with $\mathcal{H}$ as basis.*

Proof. Fist we have to prove that $\tilde{\mathcal{H}}$ is an ideal in $pq\mathrm{M}^\infty(X)$. Therefore, consider $d, e \in \tilde{\mathcal{H}}$. Then we have

$$\forall x \in X, \forall \epsilon > 0, \forall \omega < \infty : \exists d_1 \in \mathcal{H} \text{ such that } d(x,\cdot) \wedge \omega \leq d_1(x,\cdot) + \epsilon$$

and

$$\forall x \in X, \forall \epsilon > 0, \forall \omega < \infty : \exists d_2 \in \mathcal{H} \text{ such that } e(x,\cdot) \wedge \omega \leq d_2(x,\cdot) + \epsilon.$$

But then we have

$$\begin{aligned} (d \vee e)(x,\cdot) \wedge \omega &= (d(x,\cdot) \wedge \omega) \vee (e(x,\cdot) \wedge \omega) \\ &\leq (d_1(x,\cdot) + \epsilon) \vee (d_2(x,\cdot) + \epsilon) \\ &\leq (d_1 \vee d_2)(x,\cdot) + \epsilon'. \end{aligned}$$

$\mathcal{H}$ is a gauge basis, hence an ideal basis. This means that there exists $c \in \mathcal{H}$ such that $d_1 \vee d_2 \leq c$. This gives us

$$(d \vee e)(x,\cdot) \wedge \omega \leq c(x,\cdot) + \epsilon'$$

and therefore $d \vee e \in \tilde{\mathcal{H}}$. The other properties of an ideal are trivial.

Secondly we prove that $\tilde{\mathcal{H}}$ is saturated. Suppose that d is dominated by $\tilde{\mathcal{H}}$. Then we have

$$\forall x \in X, \forall \epsilon > 0, \forall \omega < \infty : \exists d_x^{\epsilon,\omega} \in \tilde{\mathcal{H}} \text{ such that } d(x,\cdot) \wedge \omega \leq d_x^{\epsilon,\omega}(x,\cdot) + \epsilon.$$

$d_x^{\epsilon,\omega} \in \tilde{\mathcal{H}}$, and therefore $d_x^{\epsilon,\omega}$ is dominated by $\mathcal{H}$. This means that we have

$$\forall y \in X, \forall \alpha > 0, \forall \beta < \infty : \exists h \in \mathcal{H} \text{ such that } d_x^{\epsilon,\omega}(y,\cdot) \wedge \beta \leq h(y,\cdot) + \alpha.$$

By combining both inequalities, we get

$$d(x,\cdot) \wedge \omega \leq h(x,\cdot) + \alpha + \epsilon,$$

and therefore d is dominated by $\mathcal{H}$. This all means that $d \in \tilde{\mathcal{H}}$ and hence $\tilde{\mathcal{H}}$ is saturated. □

Proposition 1.3.6 *If $\mathcal{H}$ is a basis for a gauge $\mathcal{G}$, then it is a gauge basis.*

Proof. If $\mathcal{H}$ is a basis for a gauge $\mathcal{G}$, then $\mathcal{H}$ is an ideal basis and $\tilde{\mathcal{H}} = \mathcal{G}$. It follows immediately from the definitions that $\mathcal{H}$ is a gauge basis. □

1.4 Towers

The last structure which we introduce is an ordered family of pretopologies on X, indexed by the real numbers in $[0, \infty]$, and fulfilling certain coherence conditions. The axioms are presented in terms of closures, but it is clear that equivalent sets of axioms can be formulated in terms of any of the defining structures of pretopological spaces, in particular neighborhood systems or convergence.

Definition 1.4.1 *A family of functions*

$$t_\epsilon : 2^X \to 2^X, \epsilon \in \mathbb{R}^+$$

is called a ***tower*** *on X, if it satisfies the following properties:*

(T1) $\forall A \in 2^X, \forall \epsilon \in \mathbb{R}^+ : A \subset t_\epsilon(A)$,

(T2) $\forall \epsilon \in \mathbb{R}^+ : t_\epsilon(\emptyset) = \emptyset$,

(T3) $\forall A, B \in 2^X, \forall \epsilon \in \mathbb{R}^+ : t_\epsilon(A \cup B) = t_\epsilon(A) \cup t_\epsilon(B)$,

(T4) $\forall A \in 2^X, \forall \epsilon, \gamma \in \mathbb{R}^+ : t_\epsilon(t_\gamma(A)) \subset t_{\epsilon+\gamma}(A)$,

(T5) $\forall A \in 2^X, \forall \epsilon \in \mathbb{R}^+ : t_\epsilon(A) = \bigcap_{\epsilon<\gamma} t_\gamma(A)$.

Notice that by (T3) and (T5) we have

$$\forall A \subset B \subset X, \forall \alpha \leq \beta \in \mathbb{R}^+ : t_\alpha(A) \subset t_\beta(B).$$

We recall that a **pretopology** on a set X is determined by an operator

$$\mathrm{cl} : 2^X \to 2^X$$

which fulfills the properties (1) $A \subset \mathrm{cl}(A)$, (2) $\mathrm{cl}(\emptyset) = \emptyset$ and (3) $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$ for all $A, B \in 2^X$. This operator is called a **pretopological closure operator**. A set X equipped with a pretopology is called a **pretopological space**.

Proposition 1.4.2 *The following are equivalent:*

1. $(t_\epsilon)_{\epsilon \in \mathbb{R}^+}$ *is a tower,*
2. *for all $\epsilon \in \mathbb{R}^+$, t_ϵ is a pretopological closure operator, and (T4) and (T5) are fulfilled.*
3. t_0 *is a topological closure operator, for all $\epsilon > 0$, t_ϵ is a pretopological closure operator, and (T4) and (T5) are fulfilled.*

Proof. Properties (T1), (T2) and (T3) constitute the definition of a pretopological closure operator. □

1.5 Approach spaces

In spite of the fact that all the concepts which were defined in section 1.1 to section 1.4 are both conceptually and technically very different from each other, we will prove that they are all equivalent. It is the purpose of this section to prove this. Not only will we prove that one type of structure unambiguously determines a unique structure of each of the other types, but we will also give several precise formulas for going from one structure to another. A structure derived from another one by such a transition will be referred to as an **associated structure**.

Since we have introduced four different structures, there are no less than twelve possible transitions. We will not describe them all, but we will restrict ourselves to those which are needed to prove the equivalences of the different structures.

The transitions which we will be describing are pictured in the following diagram.

$$\begin{array}{ccccc} & & t & & \\ & & \Updownarrow & & \\ \lambda & \Leftrightarrow & \delta & \Leftrightarrow & \mathcal{G} \end{array}$$

The following two theorems will show that distances and limit operators are equivalent concepts.

Theorem 1.5.1 ***Distance $\Rightarrow$ Limit operator***
If $\delta : X \times 2^X \to [0,\infty]$ is a distance on X, then the function

$$\lambda : \boldsymbol{F}(X) \to [0,\infty]^X : \mathcal{F} \mapsto \sup_{U \in \operatorname{sec}(\mathcal{F})} \delta_U$$

is a limit operator on X. Moreover, for any $x \in X$ and $A \in 2^X$, we have

$$\delta(x, A) = \inf_{\mathcal{U} \in \boldsymbol{U}(A)} \lambda\mathcal{U}(x).$$

Proof. (L1) follows immediately from (D1):

$$\lambda(\operatorname{stack} x)(x) = \sup_{U \in \operatorname{sec}(\operatorname{stack} x)} \delta_U(x) = 0.$$

(L2) follows from the fact that $\mathcal{F} \subset \mathcal{G}$ implies $\mathbf{U}(\mathcal{G}) \subset \mathbf{U}(\mathcal{F})$.

To prove (L3) let $(\mathcal{F}_j)_{j \in J}$ be a family of filters on X. Then, upon noticing that

$$\operatorname{sec}\Big(\bigcap_{j \in J} \mathcal{F}_j\Big) = \bigcup_{j \in J} \operatorname{sec}(\mathcal{F}_j),$$

it follows that we have

$$\begin{aligned} \lambda\Big(\bigcap_{j \in J} \mathcal{F}_j\Big) &= \sup_{U \in \operatorname{sec}\left(\bigcap_{j \in J} \mathcal{F}_j\right)} \delta_U \\ &= \sup_{U \in \bigcup_{j \in J} \operatorname{sec}(\mathcal{F}_j)} \delta_U \\ &= \sup_{j \in J} \sup_{U \in \operatorname{sec}(\mathcal{F}_j)} \delta_U \\ &= \sup_{j \in J} \lambda(\mathcal{F}_j) \end{aligned}$$

To prove (L4), let $\mathcal{F} \in \mathbf{F}(X)$ and let $\mathcal{S} : X \to \mathbf{F}(X)$ be a selection of filters on X. Let

$$\epsilon := \sup_{y \in X} \lambda(\mathcal{S}(y))(y).$$

First suppose that all filters involved are ultrafilters. For any $D \in \Sigma\mathcal{S}\mathcal{F}$, there exists an $F \in \mathcal{F}$ such that, for all $y \in F$, $D \in \mathcal{S}(y)$. Consequently,

$$\delta(y, D) \leq \lambda(\mathcal{S}(y))(y) \leq \epsilon.$$

This proves that $D^{(\epsilon)} \in \mathcal{F}$ and then it follows from (D4) that

$$\begin{aligned} \delta_D &\leq \delta_{D^{(\epsilon)}} + \epsilon \\ &\leq \sup_{F \in \mathcal{F}} \delta_F + \epsilon \\ &\leq \lambda\mathcal{F} + \epsilon. \end{aligned}$$

By arbitrariness of $D \in \Sigma\mathcal{S}\mathcal{F}$ it follows that

$$\begin{aligned} \lambda\Sigma\mathcal{S}\mathcal{F} &= \sup_{D \in \Sigma\mathcal{S}\mathcal{F}} \delta_D \\ &\leq \lambda\mathcal{F} + \epsilon. \end{aligned}$$

Secondly we consider the case where all filters involved are arbitrary. For each selection $\mathcal{R} : X \to \mathbf{U}(\mathcal{S}(y))$, let

$$\epsilon_{\mathcal{R}} := \sup_{y \in X} \lambda(\mathcal{R}(y))(y).$$

Then we have that

$$\epsilon = \sup_{\mathcal{R}: X \to \mathbf{U}(\mathcal{S}(y))} \epsilon_{\mathcal{R}}.$$

From the result for ultrafilters it then follows that, for any $\mathcal{R} : X \to \mathbf{U}(\mathcal{S}(y))$ and $\mathcal{U} \in \mathbf{U}(\mathcal{F})$

$$\lambda\Sigma\mathcal{R}\mathcal{U} \leq \lambda\mathcal{U} + \epsilon_{\mathcal{R}}.$$

It then follows from (L3) that

$$\begin{aligned} \lambda\Sigma\mathcal{S}\mathcal{F} &= \sup_{\mathcal{R}: X \to \mathbf{U}(\mathcal{S}(y))} \sup_{\mathcal{U} \in \mathbf{U}(\mathcal{F})} \lambda\Sigma\mathcal{R}\mathcal{U} \\ &\leq \sup_{\mathcal{R}: X \to \mathbf{U}(\mathcal{S}(y))} \sup_{\mathcal{U} \in \mathbf{U}(\mathcal{F})} (\lambda\mathcal{U} + \epsilon_{\mathcal{R}}) \\ &= \sup_{\mathcal{R}: X \to \mathbf{U}(\mathcal{S}(y))} (\lambda\mathcal{F} + \epsilon_{\mathcal{R}}) \\ &= \lambda\mathcal{F} + \epsilon. \end{aligned}$$

To prove the final claim of the theorem, first notice that one inequality is clear. To prove the other one, let $x \in X$ and $A \in 2^X$. It follows from the definition of λ, and upon applying complete distributivity, that

$$\begin{aligned} \inf_{\mathcal{U} \in \mathbf{U}(A)} \lambda\mathcal{U} &= \inf_{\mathcal{U} \in \mathbf{U}(A)} \sup_{U \in \mathrm{sec}(\mathcal{U})} \delta_U \\ &= \sup_{\theta \in \prod_{\mathcal{U} \in \mathbf{U}(A)} \mathcal{U}} \inf_{\mathcal{U} \in \mathbf{U}(A)} \delta_{\theta(\mathcal{U})}. \end{aligned}$$

By proposition 1.2.2 for each $\theta \in \prod_{\mathcal{U}\in\mathbf{U}(A)} \mathcal{U}$, we can find a finite subset $U_\theta \subset \mathbf{U}(A)$ such that $A \subset \bigcup_{\mathcal{U}\in U_\theta} \theta(\mathcal{U})$. Consequently, it follows from (D3) that

$$\inf_{\mathcal{U}\in\mathbf{U}(A)} \lambda\mathcal{U} \leq \sup_{\theta\in\prod_{\mathcal{U}\in\mathbf{U}(A)}\mathcal{U}} \delta_{(\bigcup_{U\in U_\theta}\theta(\mathcal{U}))} \leq \delta_A,$$

which proves the remaining inequality. □

Theorem 1.5.2 ***Limit operator*** $\Rightarrow$ ***Distance***
If $\lambda : \boldsymbol{F}(X) \to [0,\infty]^X$ is a limit operator on X, then the function

$$\delta : X \times 2^X \to [0,\infty] : (x, A) \mapsto \inf_{\mathcal{U}\in\boldsymbol{U}(A)} \lambda\mathcal{U}(x)$$

is a distance on X. Moreover, for any $\mathcal{F} \in \boldsymbol{F}(X)$ and $x \in X$, we have

$$\lambda\mathcal{F}(x) = \sup_{U\in\mathrm{sec}(\mathcal{F})} \delta(x, U).$$

Proof. (D1) follows immediately from (L1):

$$\delta(x, \{x\}) = \inf_{\mathcal{U}\in\mathbf{U}(\{x\})} \lambda\mathcal{U}(x) = \lambda(\mathrm{stack}\, x)(x) = 0.$$

(D2) is trivial:

$$\delta(x, \emptyset) = \inf_{\mathcal{U}\in\emptyset} \lambda\mathcal{U}(x) = \infty.$$

(D3) follows from the fact that, for any $A, B \in 2^X$, we have $\mathbf{U}(\mathrm{stack}(A\cup B)) = \mathbf{U}(\mathrm{stack}\, A) \cup \mathbf{U}(\mathrm{stack}\, B)$. This gives us

$$\begin{aligned}
\delta(x, A\cup B) &= \inf_{\mathcal{U}\in\mathbf{U}(\mathrm{stack}(A\cup B))} \lambda\mathcal{U}(x) \\
&= \inf_{\mathcal{U}\in\mathbf{U}(\mathrm{stack}\, A)\cup\mathbf{U}(\mathrm{stack}\, B)} \lambda\mathcal{U}(x) \\
&= \min\Big(\inf_{\mathcal{U}\in\mathbf{U}(\mathrm{stack}\, A)} \lambda\mathcal{U}(x), \inf_{\mathcal{U}\in\mathbf{U}(\mathrm{stack}\, B)} \lambda\mathcal{U}(x)\Big) \\
&= \min(\delta(x, A), \delta(x, B)).
\end{aligned}$$

We will now prove the final claim of the theorem, since we will require this in the proof of (D4). Let λ' be defined as

$$\lambda' : \mathbf{F}(X) \to [0,\infty]^X : \mathcal{F} \mapsto \sup_{U\in\mathrm{sec}(\mathcal{F})} \delta_U.$$

Let $\mathcal{U} \in \mathbf{U}(X)$, then first we have

$$\begin{aligned}
\lambda'\mathcal{U} &= \sup_{U\in\mathcal{U}} \delta_U \\
&= \sup_{U\in\mathcal{U}} \inf_{\mathcal{W}\in\mathbf{U}(U)} \lambda\mathcal{W} \\
&\leq \lambda\mathcal{U}.
\end{aligned}$$

Secondly, by complete distributivity, we have

$$\begin{aligned}\lambda'\mathcal{U} &= \sup_{U\in\mathcal{U}} \inf_{\mathcal{W}\in\mathbf{U}(U)} \lambda\mathcal{W} \\ &= \inf_{\theta\in\prod_{U\in\mathcal{U}}\mathbf{U}(U)} \sup_{U\in\mathcal{U}} \lambda(\theta(U)).\end{aligned}$$

Furthermore, for any $\theta \in \prod_{U\in\mathcal{U}} \mathbf{U}(U)$ and any $U \in \mathcal{U}$, we have $U \in \theta(U)$ and thus $\bigcap_{U\in\mathcal{U}} \theta(U) \subset \mathcal{U}$. From (L2) it then follows that

$$\lambda\mathcal{U} \leq \lambda\Big(\bigcap_{U\in\mathcal{U}} \theta(U)\Big).$$

Since this holds for all $\theta \in \prod_{U\in\mathcal{U}} \mathbf{U}(U)$, it follows that

$$\lambda\mathcal{U} \leq \lambda'\mathcal{U}.$$

Consequently λ and λ' coincide on ultrafilters. From the definition of λ' and the fact that λ fulfills (L3), this, however, suffices to conclude $\lambda = \lambda'$.

In order now to prove (D4), let $A \in 2^X$, $\epsilon \in \mathbb{R}^+$ and $\mathcal{W} \in \mathbf{U}\big(A^{(\epsilon)}\big)$.
Claim: $\forall y \in A^{(\epsilon)}\, \exists \mathcal{S}(y) \in \mathbf{U}(A)$ such that $\lambda(\mathcal{S}(y))(y) \leq \epsilon$.
Indeed, if not then, for some $y \in A^{(\epsilon)}$ and for all $\mathcal{U} \in \mathbf{U}(A)$, we have

$$\begin{aligned}\epsilon &< \lambda\mathcal{U}(y) \\ &= \lambda'\mathcal{U}(y) \\ &= \sup_{U\in\mathcal{U}} \delta(y, U).\end{aligned}$$

Consequently, for all $\mathcal{U} \in \mathbf{U}(A)$, we can find $U_{\mathcal{U}} \in \mathcal{U}$ such that $\epsilon < \delta(y, U_{\mathcal{U}})$. By proposition 1.2.2 we can find $\mathcal{U}_1, \ldots, \mathcal{U}_n \in \mathbf{U}(A)$ such that $A \subset \bigcup_{i=1}^n U_{\mathcal{U}_i}$. It then follows from (D3) that

$$\begin{aligned}\epsilon &< \inf_{i=1}^n \delta(y, U_{\mathcal{U}_i}) \\ &= \delta\Big(y, \bigcup_{i=1}^n U_{\mathcal{U}_i}\Big) \\ &\leq \delta(y, A),\end{aligned}$$

which is in contradiction with the choice of y. This proves our claim. Consequently, for all $y \in A^{(\epsilon)}$, we can fix some $\mathcal{S}(y) \in \mathbf{U}(A)$ such that $\lambda(\mathcal{S}(y))(y) \leq \epsilon$. For $y \notin A^{(\epsilon)}$ we let $\mathcal{S}(y) := \operatorname{stack} y$. Next we let

$$\epsilon' := \sup_{y\in X} \lambda(\mathcal{S}(y))(y).$$

Since $A \in \bigcap_{y\in A^{(\epsilon)}} \mathcal{S}(y)$ it follows that $A \in \Sigma\mathcal{S}\mathcal{W}$ and consequently, by proposition 1.2.3, $\Sigma\mathcal{S}\mathcal{W} \in \mathbf{U}(A)$. From the definition of δ and from (L4) we then obtain that, for all $x \in X$

$$\begin{aligned}\delta(x, A) &\leq \lambda\Sigma\mathcal{S}\mathcal{W}(x) \\ &\leq \lambda\mathcal{W}(x) + \epsilon' \\ &\leq \lambda\mathcal{W}(x) + \epsilon.\end{aligned}$$

Since this holds for all $\mathcal{W} \in \mathbf{U}\big(A^{(\epsilon)}\big)$, it follows that

$$\delta(x, A) \leq \delta\big(x, A^{(\epsilon)}\big) + \epsilon.$$

□

We now turn to the relation between distances and gauges.

Theorem 1.5.3 ***Distance*** $\Rightarrow$ ***Gauge***
If $\delta : X \times 2^X \to [0, \infty]$ is a distance on X, then

$$\mathcal{G} := \Big\{ d \in pqM^{\infty}(X) \mid \forall A \subset X, \forall x \in X : \inf_{a \in A} d(x, a) \leq \delta(x, A) \Big\}$$

is a gauge on X.

Proof. Let $\mathcal{G}_0 \in 2^{(\mathcal{G})}, x \in X$ and $A \subset X$. Then, by complete distributivity, we have that

$$\inf_{a \in A} \sup_{d \in \mathcal{G}_0} d(x, a) = \sup_{\varphi \in \mathcal{G}_0^A} \inf_{a \in A} \varphi(a)(x, a).$$

If we fix $\varphi \in \mathcal{G}_0^A$, then it follows that

$$\begin{aligned} \inf_{a \in A} \varphi(a)(x, a) &= \inf_{d \in \mathcal{G}_0} \inf_{a \in \varphi^{-1}(d)} d(x, a) \\ &\leq \inf_{d \in \mathcal{G}_0} \delta\big(x, \varphi^{-1}(d)\big) \\ &= \delta(x, A). \end{aligned}$$

Hence it follows that $\mathcal{G}$ is closed under the formation of finite suprema. The remaining properties of an ideal are trivial.

If $d \in pq\mathrm{M}^{\infty}(X)$ is such that, for all $x \in X, \epsilon > 0$ and $\omega < \infty$, there exists $d' \in \mathcal{G}$ such that $d(x, \cdot) \wedge \omega \leq d'(x, \cdot) + \epsilon$, then it follows at once that, for any $\epsilon > 0$ and $\omega < \infty$,

$$\begin{aligned} \inf_{a \in A} d(x, a) \wedge \omega &\leq \inf_{a \in A} d'(x, a) + \epsilon \\ &\leq \delta(x, A) + \epsilon. \end{aligned}$$

By arbitrariness of ϵ and ω this proves that $d \in \mathcal{G}$. Hence $\mathcal{G}$ is saturated. □

In order to be able to work with the gauge associated with a distance, as given by the foregoing theorem, it will be useful to have a specified set of ∞pq-metrics in that gauge. The following proposition provides us with such a collection.

Proposition 1.5.4 *If $\delta : X \times 2^X \to [0, \infty]$ is a distance on X, then, for any $\zeta < \infty$ and $Z \subset X$, the function*

$$d_Z^{\zeta} : X \times X \to [0, \infty] : (x, y) \mapsto \big(\delta(x, Z) \wedge \zeta - \delta(y, Z) \wedge \zeta\big) \vee 0$$

is a pq-metric in the gauge associated with δ.

Proof. That d_Z^ζ is a pq-metric is easily seen. Let $x \in X$ and $A \subset X$. Then it follows from proposition 1.1.2 that

$$\begin{aligned}
\inf_{a\in A} d_Z^\zeta(x,a) &= \inf_{a\in A} \big(\delta(x,Z)\wedge\zeta - \delta(a,Z)\wedge\zeta\big)\vee 0 \\
&\leq \big(\delta(x,Z)\wedge\zeta - \sup_{a\in A}\delta(a,Z)\wedge\zeta\big)\vee 0 \\
&\leq \Big(\big(\delta(x,A)+\sup_{a\in A}\delta(a,Z)\big)\wedge\zeta - \sup_{a\in A}\delta(a,Z)\wedge\zeta\Big)\vee 0 \\
&\leq \big(\delta(x,A)+\sup_{a\in A}\delta(a,Z)\wedge\zeta - \sup_{a\in A}\delta(a,Z)\wedge\zeta\big)\vee 0 \\
&= \delta(x,A).
\end{aligned}$$

By the characterization given in theorem 1.5.3 this proves that d_Z^ζ is indeed a member of the gauge associated with δ. □

Theorem 1.5.5 ***Gauge*** $\Rightarrow$ ***Distance***
If $\mathcal{G} \subset pqM^\infty(X)$ is a gauge basis on X, then the function

$$\delta : X \times 2^X \to [0,\infty] : (x,A) \mapsto \sup_{d\in\mathcal{G}} \inf_{a\in A} d(x,a)$$

is a distance on X.

*Proof.*Verification of (D1), (D2) and (D3) are straightforward.
(D1):

$$\begin{aligned}
\delta(x,\{x\}) &= \sup_{d\in\mathcal{G}} \inf_{a\in\{x\}} d(x,a) \\
&= \sup_{d\in\mathcal{G}} d(x,x) \\
&= 0.
\end{aligned}$$

(D2):

$$\delta(x,\emptyset) = \sup_{d\in\mathcal{G}} \inf_{a\in\emptyset} d(x,a) = \infty.$$

(D3)

$$\begin{aligned}
\delta(x,A\cup B) &= \sup_{d\in\mathcal{G}} \inf_{a\in A\cup B} d(x,a) \\
&= \min\Big(\sup_{d\in\mathcal{G}} \inf_{a\in A} d(x,a), \sup_{d\in\mathcal{G}} \inf_{a\in B} d(x,a)\Big) \\
&= \min(\delta(x,A),\delta(x,B)).
\end{aligned}$$

To show (D4) let $x \in X, A \subset X$ and $\epsilon > 0$ be fixed. Then, for any $b \in A^{(\epsilon)}$, $d \in \mathcal{G}$ and $\theta > 0$, there exists $a_d \in A$ such that $d(b,a_d) \leq \epsilon + \theta$. Consequently,

$$\begin{aligned}
d(x,a_d) &\leq d(x,b) + d(b,a_d) \\
&\leq d(x,b) + \epsilon + \theta,
\end{aligned}$$

which proves that $\inf_{a\in A} d(x,a) \leq \inf_{b\in A^{(\epsilon)}} d(x,b) + \epsilon + \theta$. Since this holds for all $d \in \mathcal{G}$, it follows that $\delta(x,A) \leq \delta\big(x,A^{(\epsilon)}\big) + \epsilon$. □

Theorem 1.5.6 *If $\delta : X \times 2^X \to [0, \infty]$ is a distance on X and $\mathcal{G}$ is the associated gauge, then we have*

$$\delta(x, A) = \sup_{d \in \mathcal{G}} \inf_{a \in A} d(x, a), \qquad \forall x \in X, \forall A \subset X.$$

Proof. Let $x \in X$ and $A \subset X$. It follows at once from theorem 1.5.3 that

$$\sup_{d \in \mathcal{G}} \inf_{a \in A} d(x, a) \leq \delta(x, A).$$

Making use of proposition 1.5.4, we further obtain

$$\begin{aligned}
\sup_{d \in \mathcal{G}} \inf_{a \in A} d(x, a) &\geq \sup_{\zeta < \infty} \sup_{Z \subset X} \inf_{a \in A} d_Z^{\zeta}(x, a) \\
&\geq \sup_{\zeta < \infty} \inf_{a \in A} d_A^{\zeta}(x, a) \\
&= \sup_{\zeta < \infty} \inf_{a \in A} \big(\delta(x, A) \wedge \zeta - \delta(a, A) \wedge \zeta\big) \vee 0 \\
&= \sup_{\zeta < \infty} \inf_{a \in A} \delta(x, A) \wedge \zeta \\
&= \delta(x, A),
\end{aligned}$$

which proves the other inequality. □

Theorem 1.5.7 *If $\mathcal{G} \subset pqM^{\infty}(X)$ is a gauge on X and δ is the associated distance, then*

$$\mathcal{G} = \Big\{ d \in pqM^{\infty}(X) \mid \forall A \subset X, \forall x \in X : \inf_{a \in A} d(x, a) \leq \delta(x, A) \Big\}.$$

Proof. Let us put

$$\mathcal{G}' := \Big\{ d \in pq\mathrm{M}^{\infty}(X) \mid \forall A \subset X, \forall x \in X : \inf_{a \in A} d(x, a) \leq \delta(x, A) \Big\}.$$

Then we know from theorem 1.5.3 that $\mathcal{G}'$ is the gauge associated with δ. That $\mathcal{G} \subset \mathcal{G}'$ follows at once from theorem 1.5.5. Now suppose that there exists $d' \in \mathcal{G}' \setminus \mathcal{G}$, then d' is not dominated by $\mathcal{G}$ and hence we can find $x \in X, \epsilon > 0$ and $\omega < \infty$ such that, for all $d \in \mathcal{G}$

$$d'(x, \cdot) \wedge \omega \not\leq d(x, \cdot) + \epsilon.$$

For each $d \in \mathcal{G}$, let $A(d) := \{y \in X \mid d'(x, y) \wedge \omega > d(x, y) + \epsilon\}$. Then it is clear that, for all $d, e \in \mathcal{G}$, $A(d) \cap A(e) = A(d \vee e) \neq \emptyset$. Hence, making use of theorem 1.5.6, we obtain

$$
\begin{aligned}
\sup_{d\in\mathcal{G}} \delta\big(x, A(d)\big) \wedge \omega &= \sup_{d\in\mathcal{G}} \sup_{e\in\mathcal{G}} \inf_{y\in A(d)} e(x,y) \wedge \omega \\
&\leq \sup_{d\in\mathcal{G}} \sup_{e\in\mathcal{G}} \inf_{y\in A(d\vee e)} (d\vee e)(x,y) \wedge \omega \\
&= \sup_{d\in\mathcal{G}} \inf_{y\in A(d)} d(x,y) \wedge \omega \\
&\leq \sup_{d\in\mathcal{G}} \inf_{y\in A(d)} \big(d'(x,y) \wedge \omega - \epsilon\big) \wedge \omega \\
&\leq \sup_{d\in\mathcal{G}} \inf_{y\in A(d)} d'(x,y) \wedge \omega - \epsilon \\
&\leq \sup_{d\in\mathcal{G}} \sup_{d''\in\mathcal{G}'} \inf_{y\in A(d)} d''(x,y) \wedge \omega - \epsilon \\
&= \sup_{d\in\mathcal{G}} \delta\big(x, A(d)\big) \wedge \omega - \epsilon.
\end{aligned}
$$

which is a contradiction. □

The combined results of 1.5.3, 1.5.4, 1.5.5, 1.5.6 and 1.5.7 prove that distances and gauges are equivalent systems.

We end our comparison of distances and gauges with the result that if a collection of ∞pq-metrics produces a formula for a distance, as in theorem 1.5.5, then it must be a basis of the associated gauge.

Theorem 1.5.8 *If δ is a distance on X and $\mathcal{G} \subset pqM^{\infty}(X)$ is a collection of ∞pq-metrics such that, for all $x \in X$ and $A \subset X$, we have*

$$\delta(x, A) = \sup_{d\in\mathcal{D}} \inf_{a\in A} d(x,a),$$

then $\mathcal{D}$ is a basis for the gauge $\mathcal{G}$ associated with δ.

Proof. It follows from theorem 1.5.7 that $\mathcal{D} \subset \mathcal{G}$. Suppose that there exists $d_0 \in \mathcal{G} \setminus \tilde{\mathcal{D}}$. Then d_0 is not dominated by $\mathcal{D}$ and hence there exists $x \in X, \epsilon > 0$ and $\omega < \infty$ such that for all $d \in \mathcal{D}$,

$$d_0(x,\cdot) \wedge \omega \not\leq \underset{\cdot}{(x,\cdot)} + \epsilon.$$

For each $d \in \mathcal{D}$, let $A(d) := \{y \in X \mid d_0(x,y) \wedge \omega > d(x,y) + \epsilon\}$. Then it is clear that, for all $d, e \in \mathcal{G}$, $A(d) \cap A(e) = A(d \vee e) \neq \emptyset$. Hence, making use of theorem 1.5.6, we obtain

$$
\begin{aligned}
\sup_{d\in\mathcal{D}} \delta\big(x, A(d)\big) \wedge \omega &= \sup_{d\in\mathcal{D}} \sup_{e\in\mathcal{D}} \inf_{y\in A(d)} e(x,y) \wedge \omega \\
&\leq \sup_{d\in\mathcal{D}} \sup_{e\in\mathcal{D}} \inf_{y\in A(d\vee e)} (d\vee e)(x,y) \wedge \omega \\
&= \sup_{d\in\mathcal{D}} \inf_{y\in A(d)} d(x,y) \wedge \omega \\
&\leq \sup_{d\in\mathcal{D}} \inf_{y\in A(d)} \big(d_0(x,y) \wedge \omega - \epsilon\big) \wedge \omega \\
&\leq \sup_{d\in\mathcal{D}} \inf_{y\in A(d)} d_0(x,y) \wedge \omega - \epsilon \\
&\leq \sup_{d\in\mathcal{D}} \sup_{e\in\mathcal{D}} \inf_{y\in A(d)} e(x,y) \wedge \omega - \epsilon \\
&= \sup_{d\in\mathcal{D}} \delta\big(x, A(d)\big) \wedge \omega - \epsilon
\end{aligned}
$$

which is a contradiction. $\square$

In the following corollary, given $\zeta < \infty$ and $Z \subset X$, d_Z^ζ stands for the ∞pq-metric defined in proposition 1.5.4.

Corollary 1.5.9 *If δ is a distance on X, then the family of pq-metrics*

$$\mathcal{H} := \left\{ d_Z^\zeta \mid \zeta < \infty, Z \in 2^X \right\}$$

is a basis for the gauge associated with δ.

Proof. This follows from the foregoing result and from the proof of theorem 1.5.6, where it was implicitly shown that $\delta(x, A) = \sup_{d \in \mathcal{H}} \inf_{a \in A} d(x, a)$. $\square$

Finally we study the relationship between distances and towers.

Theorem 1.5.10 ***Distance $\Rightarrow$ Tower***
If $\delta : X \times 2^X \to [0, \infty]$ is a distance on X, then the family $(t_\epsilon)_{\epsilon \in \mathbb{R}^+}$ defined by

$$t_\epsilon(A) := A^{(\epsilon)} \qquad \forall A \subset X, \forall \epsilon \in \mathbb{R}^+,$$

is a tower on X. Moreover, for any $x \in X$ and $A \subset X$, we have

$$\delta(x, A) = \inf\{\epsilon \in \mathbb{R}^+ \mid x \in t_\epsilon(A)\}.$$

Proof. (T1), (T2) and (T3) follow immediately from (D1), (D2) and (D3).
(T1): For all $A \subset X$ we have: $A \subset A^{(\epsilon)} = t_\epsilon(A)$.
(T2): $t_\epsilon(\emptyset) = \emptyset^{(\epsilon)} = \{x \in X \mid \delta(x, \emptyset) \leq \epsilon\} = \emptyset$.
(T3): Let $A, B \in 2^X$, then we have

$$\begin{aligned} t_\epsilon(A \cup B) &= (A \cup B)^{(\epsilon)} \\ &= \{x \in X \mid \delta(x, A \cup B) \leq \epsilon\} \\ &= \{x \in X \mid \min(\delta(x, A), \delta(x, B)) \leq \epsilon\} \\ &= \{x \in X \mid \delta(x, A) \leq \epsilon\} \cup \{x \in X \mid \delta(x, B) \leq \epsilon\} \\ &= A^{(\epsilon)} \cup B^{(\epsilon)} \\ &= t_\epsilon(A) \cup t_\epsilon(B). \end{aligned}$$

We now prove (T4). Let $A \subset X, \epsilon, \gamma \in \mathbb{R}^+$.

$$\begin{aligned} x \in t_\epsilon(t_\gamma(A)) &\Rightarrow \delta\big(x, A^{(\gamma)}\big) \leq \epsilon \\ &\Rightarrow \delta(x, A) \leq \epsilon + \gamma \\ &\Rightarrow x \in t_{\epsilon+\gamma}(A). \end{aligned}$$

(T5) follows immediately from the fact that $\delta(x, A) \leq \epsilon$ if and only if, for all $\epsilon' > \epsilon, \delta(x, A) \leq \epsilon'$.

The last claim from the theorem follows at once from the definitions. $\square$

Theorem 1.5.11 ***Tower $\Rightarrow$ Distance***
If $(t_\epsilon)_{\epsilon \in \mathbb{R}^+}$ is a tower on X, then the function

$$\delta : X \times 2^X \to [0, \infty]$$

defined by

$$\delta(x, A) := \inf\{\epsilon \in \mathbb{R}^+ \mid x \in t_\epsilon(A)\}$$

is a distance on X. Moreover, we have that

$$t_\epsilon(A) = A^{(\epsilon)}, \qquad \forall \epsilon \in \mathbb{R}^+, \forall A \subset X.$$

Proof. (D1), (D2) and (D3) follow at once from (T1), (T2) and (T3).
(D1): $\delta(x, \{x\}) = \inf\{\epsilon \in \mathbb{R}^+ \mid x \in t_\epsilon(\{x\})\} = 0$.
(D2): $\delta(x, \emptyset) = \inf\{\epsilon \in \mathbb{R}^+ \mid x \in t_\epsilon(\emptyset)\} = \inf\{\epsilon \in \mathbb{R}^+ \mid x \in \emptyset\} = \inf \emptyset = \infty$.
(D3): Let $A, B \in 2^X$, then we have

$$\begin{aligned}
\delta(x, A \cup B) &= \inf\{\epsilon \in \mathbb{R}^+ \mid x \in t_\epsilon(A \cup B)\} \\
&= \inf\{\epsilon \in \mathbb{R}^+ \mid x \in t_\epsilon(A) \cup t_\epsilon(B)\} \\
&= \min(\inf\{\epsilon \in \mathbb{R}^+ \mid x \in t_\epsilon(A)\}, \inf\{\epsilon \in \mathbb{R}^+ \mid x \in t_\epsilon(B)\}) \\
&= \min(\delta(x, A), \delta(x, B)).
\end{aligned}$$

To prove (D4) we first prove the last claim of the theorem. Let $x \in X, A \subset X$, and $\epsilon \in \mathbb{R}^+$, then we have

$$\begin{aligned}
x \in A^{(\epsilon)} &\Rightarrow \inf\{\alpha \in \mathbb{R}^+ \mid x \in t_\alpha(A)\} \leq \epsilon \\
&\Rightarrow \forall \alpha > \epsilon : x \in t_\alpha(A) \\
&\Rightarrow x \in \bigcap_{\alpha > \epsilon} t_\alpha(A) = t_\epsilon(A).
\end{aligned}$$

Conversely, we have

$$\begin{aligned}
x \in t_\epsilon(A) &\Rightarrow \delta(x, A) = \inf\{\alpha \in \mathbb{R}^+ \mid x \in t_\alpha(A)\} \leq \epsilon \\
&\Rightarrow x \in A^{(\epsilon)}.
\end{aligned}$$

(D4) now follows from the observation that if $\delta\big(x, A^{(\epsilon)}\big) < \alpha$ then $x \in t_\alpha\big(A^{(\epsilon)}\big) = t_\alpha\big(t_\epsilon(A)\big) \subset t_{\alpha+\epsilon}(A)$ and thus $\delta(x, A) \leq \alpha + \epsilon$. □

We will also introduce the transition formula from gauges to limit operators. This will be useful in following properties.

Theorem 1.5.12 ***Gauge $\Rightarrow$ Limit operator***
If $\mathcal{G} \subset pqM^\infty(X)$ is a gauge on X, then the associated limit operator is given by

$$\lambda\mathcal{F}(x) = \sup_{d \in \mathcal{G}} \inf_{F \in \mathcal{F}} \sup_{y \in F} d(x, y) \qquad \forall x \in X, \forall \mathcal{F} \in \boldsymbol{F}(X).$$

Definition 1.5.13 *A pair $(X, \mathfrak{G})$, where $\mathfrak{G}$ is a distance, a limit operator, a gauge or a tower, is called an* ***approach space****.*

As we have just seen, an approach space can be determined by giving any of these equivalent structures. If no confusion is possible, we will always denote a distance by δ and the associated limit operator, gauge and tower by λ, $\mathcal{G}$ and $t := (t_\epsilon)_{\epsilon \in \mathbb{R}^+}$. If we are dealing with different approach spaces at the same time, we will make clear which structures are associated with each other, by using appropriate indices or accents.

When an approach space is determined by a gauge basis, then we will usually refer to the other structures of the approach space as being generated by that basis.

We use both notations $A^{(\epsilon)}$ and $t_\epsilon(A)$ for $\{x \in X \mid \delta(x, A) \leq \epsilon\}$. In all that follows we will freely use the different structures defining approach spaces as well as the transitions among them. At the end of this section, for easy reference, we give some tables containing the most important transition formulas.

Now we will introduce two special examples. They both have $[0, \infty]$ as the underlying set. We denote by $d_{\mathbb{E}}$ the ∞-metric defined by

$$d_{\mathbb{E}} : [0, \infty] \times [0, \infty] \to [0, \infty] : (x, y) \to |x - y|,$$

where

$$\begin{array}{l} d_{\mathbb{E}}(x, \infty) = d_{\mathbb{E}}(\infty, x) = \infty \quad \forall x \in X \\ \text{and} \\ d_{\mathbb{E}}(\infty, \infty) = 0 \end{array}$$

and by $d_{\mathbb{P}}$ the ∞q-metric defined by

$$d_{\mathbb{P}} : [0, \infty] \times [0, \infty] \to [0, \infty] : (x, y) \to (x - y) \vee 0,$$

where

$$\begin{array}{ll} d_{\mathbb{P}}(x, \infty) = 0 & \forall x \in X, \\ d_{\mathbb{P}}(\infty, x) = \infty & \forall x \in X, \\ \text{and} & \\ d_{\mathbb{P}}(\infty, \infty) = 0 & \end{array}$$

Note that both are straightforward extensions of well-known ∞pq-metrics on $[0, \infty[$. The restriction of $d_{\mathbb{E}}$ to $[0, \infty[$ will be denoted by d_E (this is the usual Euclidean metric) and the restriction of $d_{\mathbb{P}}$ to $[0, \infty[$ will be denoted by d_P. For the topologies generated by $d_{\mathbb{E}}$ and by $d_{\mathbb{P}}$ the point ∞ is isolated.

Examples 1.5.14

1. Define

$$\delta_{\mathbb{E}} : [0, \infty] \times 2^{[0,\infty]} \to [0, \infty]$$

by

$$\delta_{\mathbb{E}}(x, A) := \begin{cases} 0 & x = \infty, A \text{ unbounded}, \\ \infty & x = \infty, A \text{ bounded}, \\ \inf_{a \in A} |x - a| & x < \infty. \end{cases}$$

We verify that this function is a distance on $[0, \infty]$.
(D1): Suppose $x < \infty$, then $\delta_{\mathbb{E}}(x, \{x\}) = \inf_{a \in \{x\}} |x - a| = |x - x| = 0$. If $x = \infty$, we get $\delta_{\mathbb{E}}(\infty, \{\infty\}) = 0$.
(D2): Let $x < \infty$, then $\delta_{\mathbb{E}}(x, \emptyset) = \inf \emptyset = \infty$. If $x = \infty$, we get $\delta_{\mathbb{E}}(\infty, \emptyset) = \infty$.
(D3): First suppose that $x < \infty$. Then we get

$$\begin{aligned} \delta_{\mathbb{E}}(x, A \cup B) &= \inf_{a \in A \cup B} |x - a| \\ &= \min\big(\inf_{a \in A} |x - a|, \inf_{a \in B} |x - a|\big) \\ &= \min\big(\delta_{\mathbb{E}}(x, A), \delta_{\mathbb{E}}(x, B)\big). \end{aligned}$$

If $x = \infty$, we have two possibilities. First we assume that both $A \subset [0, \infty]$ and $B \subset [0, \infty]$ are bounded. Then so is $A \cup B$. Therefore we get

$$\delta_{\mathbb{E}}(\infty, A \cup B) = \infty = \min(\delta_{\mathbb{E}}(\infty, A), \delta_{\mathbb{E}}(\infty, B)).$$

Consider now the possibility that $A \subset [0, \infty]$ or $B \subset [0, \infty]$ is unbounded. Then $A \cup B$ is unbounded as well, and we have

$$\delta_{\mathbb{E}}(\infty, A \cup B) = 0 = \min(\delta_{\mathbb{E}}(\infty, A), \delta_{\mathbb{E}}(\infty, B))$$

(D4): If $x < \infty$, then we have

$$\delta_{\mathbb{E}}(x, A) = \inf_{a \in A} |x - a| \leq \inf_{a \in A^{(\epsilon)}} |x - a| + \epsilon = \delta_{\mathbb{E}}\big(x, A^{(\epsilon)}\big).$$

If $A \subset [0, \infty]$ is bounded, we get

$$\delta_{\mathbb{E}}(\infty, A) = 0 \leq \delta(\infty, A^{(\epsilon)}) + \epsilon.$$

If $A \subset [0, \infty]$ is unbounded, then so is $A^{(\epsilon)}$. We get that $\delta_{\mathbb{E}}(\infty, A) = \infty$ and $\delta_{\mathbb{E}}\big(\infty, A^{(\epsilon)}\big) = \infty$, and thus $\delta_{\mathbb{E}}(\infty, A) \leq \delta_{\mathbb{E}}\big(\infty, A^{(\epsilon)}\big) + \epsilon$.

We will denote the approach space thus obtained by

$$\mathbb{E} := ([0, \infty], \delta_{\mathbb{E}}).$$

This distance actually extends the Euclidean metric d_E to the Alexandroff compactification of $[0, \infty[$. The space $[0, \infty]$ equipped with the Alexandroff compactification is of course metrizable, but since the Euclidean metric on $[0, \infty[$ is unbounded, this topology can of course not be metrized by a metric which extends the Euclidean metric. This, however, is precisely what the distance $\delta_{\mathbb{E}}$ does. For $x \in [0, \infty[$ and $A \subset [0, \infty[$ we have that $\delta_{\mathbb{E}}(x, A) = \inf_{x \in A} d_E(x, a)$, and so, on $[0, \infty[$, $\delta_{\mathbb{E}}$ is completely determined by d_E.

2. Define

$$\delta_{\mathbb{P}} : [0, \infty] \times 2^{[0,\infty]} \to [0, \infty]$$

by

$$\delta_{\mathbb{P}}(x, A) := \begin{cases} (x - \sup A) \vee 0 & A \neq \emptyset \\ \infty & A = \emptyset. \end{cases}$$

It is easy to see that this function is also a distance on $[0, \infty]$. We will denote the approach space thus defined by

$$\mathbb{P} := ([0, \infty], \delta_{\mathbb{P}}).$$

The limit operator associated with this distance is determined as follows. Given any filter $\mathcal{F}$ on $[0, \infty]$, let

$$l(\mathcal{F}) := \inf_{U \in \mathrm{sec}(\mathcal{F})} \sup U.$$

Then we have for all $x \in [0, \infty]$:

$$\begin{aligned}
\lambda_{\mathbb{P}}\mathcal{F}(x) &= \sup_{A \in \mathrm{sec}(\mathcal{F})} \delta_{\mathbb{P}}(x, A) \\
&= \sup_{A \in \mathrm{sec}(\mathcal{F})} \big(x - \sup A\big) \vee 0 \\
&= \big(x - \inf_{A \in \mathrm{sec}(\mathcal{F})} \sup A\big) \vee 0 \\
&= \big(x - l(\mathcal{F})\big) \vee 0.
\end{aligned}$$

Notice that for any filter $\mathcal{F}$, either $\lambda_{\mathbb{P}}\mathcal{F}(\infty) = 0$ or $\lambda_{\mathbb{P}}\mathcal{F}(\infty) = \infty$. The first case will occur if $l(\mathcal{F}) = \infty$, i.e. if every member of $\mathrm{sec}(\mathcal{F})$ is unbounded.

The gauge associated with the distance $\delta_{\mathbb{P}}$ is the gauge with basis

$$\mathcal{H} = \{d_\alpha \mid \alpha \in [0, \infty[\}$$

where

$$d_\alpha(x, y) = \big((x \wedge \alpha) - (y \wedge \alpha)\big) \vee 0.$$

Consider the distance δ, associated with the gauge $\mathcal{G} = \tilde{\mathcal{H}}$. Then we have for every $x \in [0, \infty]$ and $A \subset [0, \infty]$

$$\begin{aligned}
\delta(x, A) &= \sup_{d_\alpha \in \mathcal{H}} \inf_{y \in A} d_\alpha(x, y) \\
&= \sup_{\alpha \in [0, \infty[} \inf_{y \in A} \Big(\big((x \wedge \alpha) - (y \wedge \alpha)\big) \vee 0\Big) \\
&= \sup_{\alpha \in [0, \infty[} \big((x \wedge \alpha) - \sup_{y \in A}(y \wedge \alpha)\big) \vee 0 \\
&= \sup_{\alpha \in [0, \infty[} \delta_{\mathbb{P}}(x \wedge \alpha, A \wedge \alpha) \\
&= \delta_{\mathbb{P}}(x, A).
\end{aligned}$$

There is still another structure which can be associated with an approach space and which could actually also be axiomatized in order to give an equivalent defining structure. However, we will not be interested in doing this here. We do nevertheless need to be able to use this structure in what follows and hence we now define it in terms of different approach structures.

Definition 1.5.15 *Let (X, δ) be an approach space. We define the* ***adherence operator*** *(associated with δ and with all the other defining structures) as the function*

$$\alpha : \boldsymbol{F}(X) \to [0, \infty]^X$$

determined by

$$\alpha\mathcal{F}(x) := \sup_{F \in \mathcal{F}} \delta(x, F) \quad \forall x \in X, \forall \mathcal{F} \in \boldsymbol{F}(X).$$

The function

$$\alpha\mathcal{F} : X \to [0, \infty]$$

is called the ***adherence*** *of $\mathcal{F}$.*

The interpretation of this operator is analogous to that of a limit operator. In each point $x \in X$, the value $\alpha\mathcal{F}(x)$ indicates how far the point x is away from being an adherence point of the filter $\mathcal{F}$.

A justification of these interpretations is given in the following results, where we discover some basic relations between the concepts of limit operator and adherence operator which are very similar to the relations which exist between the notions of adherence point and limit point of a filter.

Proposition 1.5.16 *Let (X, δ) be an approach space. Then for all $\mathcal{F}, \mathcal{G} \in \boldsymbol{F}(X)$ we have*

$$\mathcal{F} \subset \mathcal{G} \Rightarrow \alpha\mathcal{F} \leq \alpha\mathcal{G} \leq \lambda\mathcal{G} \leq \lambda\mathcal{F}.$$

Proof. This follows immediately from theorem 1.5.1 and the definition of the adherence operator in 1.5.15. □

Notice that in topological spaces, finer filters converge to more and adhere to fewer points. Comparing this to the result of proposition 1.5.16 gives a first argument justifying our interpretation of the value of the limit and adherence operator of a filter.

The following result generalizes the well-known fact that for ultrafilters limit points and adherence points are the same.

Proposition 1.5.17 *Let (X, δ) be an approach space. Then, for all $\mathcal{U} \in \boldsymbol{U}(X)$, we have*

$$\lambda\mathcal{U} = \alpha\mathcal{U}.$$

Proof. If $\mathcal{U}$ is an ultrafilter, then $\sec(\mathcal{U}) = \mathcal{U}$. Using theorem 1.5.1, we get

$$\alpha\mathcal{U} = \sup_{U \in \mathcal{U}} \delta_U = \sup_{U \in \sec(\mathcal{U})} \delta_U = \lambda\mathcal{U}.$$

□

Proposition 1.5.18 *Let (X, δ) be an approach space. Then for all $\mathcal{F} \in \boldsymbol{F}(X)$ and $x \in X$, we have*

1. $\lambda\mathcal{F}(x) = \sup_{\mathcal{U} \in \boldsymbol{U}(\mathcal{F})} \lambda\mathcal{U}(x) = \sup_{\mathcal{U} \in \boldsymbol{U}(\mathcal{F})} \alpha\mathcal{U}(x)$,
2. $\alpha\mathcal{F}(x) = \inf_{\mathcal{U} \in \boldsymbol{U}(\mathcal{F})} \lambda\mathcal{U}(x) = \inf_{\mathcal{U} \in \boldsymbol{U}(\mathcal{F})} \alpha\mathcal{U}(x)$.

Proof. 1. This follows from theorem 1.5.1 and proposition 1.5.17.
2. By complete distributivity, we have

$$\begin{aligned} \inf_{\mathcal{U} \in \mathbf{U}(\mathcal{F})} \lambda\mathcal{U}(x) &= \inf_{\mathcal{U} \in \mathbf{U}(\mathcal{F})} \sup_{U \in \mathcal{U}} \delta(x, U) \\ &= \sup_{\sigma \in \prod_{\mathcal{U} \in \mathbf{U}(\mathcal{F})} \mathcal{U}} \inf_{\mathcal{U} \in \mathbf{U}(\mathcal{F})} \delta\big(x, \sigma(\mathcal{U})\big). \end{aligned}$$

From proposition 1.2.2 it follows that, for any $\sigma \in \prod_{\mathcal{U}\in\mathbf{U}(\mathcal{F})}\mathcal{U}$, we can find a finite subcollection $U_\sigma \subset \mathbf{U}(\mathcal{F})$ such that

$$\bigcup_{\mathcal{U}\in U_\sigma} \sigma(\mathcal{U}) \in \mathcal{F}.$$

Making use of this, it follows that

$$\begin{aligned}
\inf_{\mathcal{U}\in\mathbf{U}(\mathcal{F})} \lambda\mathcal{U}(x) &\leq \sup_{\sigma\in\prod_{\mathcal{U}\in\mathbf{U}(\mathcal{F})}\mathcal{U}} \inf_{\mathcal{U}\in U_\sigma} \delta\big(x, \sigma(\mathcal{U})\big) \\
&= \sup_{\sigma\in\prod_{\mathcal{U}\in\mathbf{U}(\mathcal{F})}\mathcal{U}} \delta\Big(x, \bigcup_{\mathcal{U}\in U_\sigma} \sigma(U)\Big) \\
&\leq \alpha\mathcal{F}(x).
\end{aligned}$$

The other inequality is immediate. □

The following result generalizes the fact that in a topological space a filter converges to a point if and only if all ultrafilters which are finer converge to that point and that a filter adheres to a point if and only if there exists an ultrafilter which is finer and which converges to that point.

Proposition 1.5.19 *Let (X, δ) be an approach space. For all $\mathcal{F} \in \boldsymbol{F}(X)$ and $x \in X$ there exists $\mathcal{U} \in \boldsymbol{U}(\mathcal{F})$ such that $\alpha\mathcal{F}(x) = \lambda\mathcal{U}(x)$.*

Proof. Let $x \in X$ and suppose on the contrary that, for all $\mathcal{U} \in \mathbf{U}(\mathcal{F})$, $\alpha\mathcal{F}(x) < \lambda\mathcal{U}(x)$. Then it follows from proposition 1.5.17 and the definition of the adherence operator that, for each $\mathcal{U} \in \mathbf{U}(\mathcal{F})$ there exists $U_\mathcal{U} \in \mathcal{U}$ such that $\alpha\mathcal{F}(x) < \delta(x, U_\mathcal{U})$. It now follows from proposition 1.2.2 that there exists a finite subset $U_0 \subset \mathbf{U}(\mathcal{F})$ such that $\bigcup_{\mathcal{U}\in U_0} U_\mathcal{U} \in \mathcal{F}$ and consequently, by the definition of the adherence operator,

$$\begin{aligned}
\alpha\mathcal{F}(x) &\geq \delta\Big(x, \bigcup_{\mathcal{U}\in U_0} U_\mathcal{U}\Big) \\
&= \min_{\mathcal{U}\in U_0} \delta(x, U_\mathcal{U}) \\
&> \alpha\mathcal{F}(x),
\end{aligned}$$

which is a contradiction. □

The following characterization of the adherence operator in terms of the gauge will also be of use.

Proposition 1.5.20 *Let (X, δ) be an approach space. Then for all $\mathcal{F} \in \boldsymbol{F}(X)$ and $x \in X$, we have*

$$\alpha\mathcal{F}(x) = \sup_{d\in\mathcal{G}} \sup_{F\in\mathcal{F}} \inf_{y\in F} d(x, y).$$

Proof. This follows from theorem 1.5.5 and the definition of the adherence operator in 1.5.15. □

Proposition 1.5.21 *Let (X, δ) be an approach space. Then for all $\mathcal{F} \in \boldsymbol{F}(X)$ and $x, y \in X$, we have*

$$\lambda\mathcal{F}(x) \leq \delta(x, \{y\}) + \lambda\mathcal{F}(y).$$

Proof. Let $x, y \in X$. By theorem 1.5.12 we get

$$\begin{array}{rcl} \lambda\mathcal{F}(x) & = & \sup_{d\in\mathcal{G}} \inf_{F\in\mathcal{F}} \sup_{z\in F} d(x,z) \\ & \leq & \sup_{d\in\mathcal{G}} \inf_{F\in\mathcal{F}} \sup_{z\in F} (d(x,y) + d(y,z)) \\ & \leq & \sup_{d\in\mathcal{G}} d(x,y) + \sup_{d\in\mathcal{G}} \inf_{F\in\mathcal{F}} \sup_{z\in F} d(y,z) \\ & = & \delta(x,\{y\}) + \lambda\mathcal{F}(y). \end{array}$$

□

To conclude this section, for easy reference, we recall the transition formulas which we proved. The tables also contain some formulas which we did not prove. These transitions, however, are easy to prove by making use of the ones we did show.

Transition formulas from a distance δ

Limit operator (1.5.1)	$\lambda\mathcal{F}(x) = \sup_{A\in\mathrm{sec}(\mathcal{F})} \delta(x, A)$
Adherence operator (1.5.15)	$\alpha\mathcal{F}(x) = \sup_{F\in\mathcal{F}} \delta(x, F)$
Gauge (1.5.3)	$\mathcal{G} = \left\{ d \in pq\mathrm{M}^\infty(X) \mid \forall A \subset X : \inf_{a\in A} d(\cdot, a) \leq \delta_A \right\}$ or $\mathcal{G} = \{d \in pq\mathrm{M}^\infty(X) \mid \delta_d \leq \delta\}$ (3.1.7)
Tower (1.5.10)	$t_\epsilon(A) = A^{(\epsilon)} = \{x \in X \mid \delta(x, A) \leq \epsilon\}$

Transition formulas from a limit operator λ

Distance (1.5.2)	$\delta(x, A) = \inf_{\mathcal{U}\in\mathbf{U}(A)} \lambda\mathcal{U}(x)$
Adherence operator (1.5.18)	$\alpha\mathcal{F}(x) = \inf_{\mathcal{U}\in\mathbf{U}(\mathcal{F})} \lambda\mathcal{U}(x)$
Gauge	$\mathcal{G} = \left\{ d \in pq\mathrm{M}^\infty(X) \mid \forall \mathcal{U} \in \mathbf{U}(X) : \sup_{U\in\mathcal{U}} \inf_{y\in U} d(\cdot, y) \leq \lambda\mathcal{U} \right\}$ or $\mathcal{G} = \{d \in pq\mathrm{M}^\infty(X) \mid \lambda_d \leq \lambda\}$ (3.1.7)
Tower	$t_\epsilon(A) = \{x \in X \mid \exists \mathcal{F} \in \mathbf{F}(A) : \lambda\mathcal{F}(x) \leq \epsilon\}$

Transition formulas from a gauge $\mathcal{G}$

Distance (1.5.5)	$\delta(x, A) = \sup_{d\in\mathcal{G}} \inf_{y\in A} d(x, y)$ or $\delta = \sup_{d\in\mathcal{G}} \delta_d$ (3.1.7)
Limit operator (1.5.12)	$\lambda\mathcal{F}(x) = \sup_{d\in\mathcal{G}} \inf_{F\in\mathcal{F}} \sup_{y\in F} d(x, y)$ or $\lambda = \sup_{d\in\mathcal{G}} \lambda_d$ (3.1.7)
Adherence operator (1.5.20)	$\alpha\mathcal{F}(x) = \sup_{d\in\mathcal{G}} \sup_{F\in\mathcal{F}} \inf_{y\in F} d(x, y)$ or $\alpha = \sup_{d\in\mathcal{G}} \alpha_d$ (3.1.7)

Transition formulas from a tower t

Distance (1.5.11)	$\delta(x, A) = \inf\{\epsilon \in \mathbb{R}^+ \mid x \in t_\epsilon(A)\}$
Limit operator	$\lambda\mathcal{F}(x) = \sup_{A\in\mathrm{sec}(\mathcal{F})} \inf\{\epsilon \in \mathbb{R}^+ \mid x \in t_\epsilon(A)\}$
Adherence operator	$\alpha\mathcal{F}(x) = \sup_{F\in\mathcal{F}} \inf\{\epsilon \in \mathbb{R}^+ \mid x \in t_\epsilon(F)\}$
Gauge	$\mathcal{G} = \Big\{d \in pq\mathrm{M}^\infty(X) \mid \forall A \subset X : t_\epsilon(A) \subset \big\{\inf_{y\in A} d(\cdot, y) \leq \epsilon\big\}\Big\}$ or $\mathcal{G} = \{d \in pq\mathrm{M}^\infty(X) \mid t_\epsilon \leq t_\epsilon^d\}$ (3.1.7)

1.6 Contractions

In this section we will introduce the morphisms which are associated with the structures defined in the foregoing sections. We will give the definition in terms of distances, but we will immediately show that these morphisms can be nicely characterized by all of the other approach structures as well.

Definition 1.6.1 *If (X, δ) and (X', δ') are approach spaces, then a function*

$$f : X \to X'$$

is called a ***contraction*** *if*

$$\forall x \in X, \forall A \in 2^X : \delta'(f(x), f(A)) \leq \delta(x, A).$$

Theorem 1.6.2 *If (X, δ) and (X', δ') are approach spaces and $f : X \to X'$, then the following are equivalent:*

1. *f is a contraction,*
2. *$\forall\mathcal{F} \in \boldsymbol{F}(X) : \lambda'(f(\mathcal{F})) \circ f \leq \lambda\mathcal{F}$,*

3. $\forall \mathcal{F} \in \boldsymbol{U}(X) : \lambda'(f(\mathcal{F})) \circ f \leq \lambda \mathcal{F}$,

4. $\forall d' \in \mathcal{G}' : d' \circ (f \times f) \in \mathcal{G}$,

5. $\forall A \in 2^X, \forall \epsilon \in [0, \infty] : f(t_\epsilon(A)) \subset t'_\epsilon(f(A))$.

Proof. $1 \Rightarrow 2$. For any $\mathcal{F} \in \mathbf{F}(X)$, we have

$$\begin{aligned} \lambda'(f(\mathcal{F})) \circ f &= \sup_{W \in \mathrm{sec}(f(\mathcal{F}))} \delta'_W \circ f \\ &\leq \sup_{U \in \mathrm{sec}(\mathcal{F})} \delta'_{f(U)} \circ f \\ &\leq \sup_{U \in \mathrm{sec}(\mathcal{F})} \delta_U \\ &= \lambda \mathcal{F}. \end{aligned}$$

$2 \Rightarrow 3$. This is trivial.
$3 \Rightarrow 1$. For any $A \in 2^X$, we have

$$\begin{aligned} \delta'_{f(A)} \circ f &= \inf_{\mathcal{W} \in \mathbf{U}(f(A))} \lambda' \mathcal{W} \circ f \\ &\leq \inf_{\mathcal{U} \in \mathbf{U}(A)} \lambda'(f(\mathcal{U})) \circ f \\ &\leq \inf_{\mathcal{U} \in \mathbf{U}(A)} \lambda \mathcal{U} \\ &= \delta_A. \end{aligned}$$

$1 \Rightarrow 4$. Let $d' \in \mathcal{G}'$. This means that for all $A' \subset X', \inf_{a \in A'} d'(\cdot, a) \leq \delta'_{A'}$. Let $A \subset X$, then we have for all $x \in X$

$$\begin{aligned} \inf_{a \in A} d' \circ (f \times f)(x, a) &= \inf_{a \in A} d'(f(x), f(a)) \\ &= \inf_{y \in f(A)} d'(f(x), y) \\ &\leq \delta'_{f(A)}(f(x)) \\ &\leq \delta_A(x). \end{aligned}$$

Hence, by theorem 1.5.3 we have that $d' \circ f \times f \in \mathcal{G}$.
$4 \Rightarrow 1$. Let $x \in X$ and $A \subset X$.

$$\begin{aligned} \delta'(f(x), f(A)) &= \sup_{d' \in \mathcal{G}'} \inf_{y \in f(A)} d'(f(x), y) \\ &\leq \sup_{d' \in \mathcal{G}'} \inf_{z \in A} d'(f(x), f(z)) \\ &= \sup_{d' \in \mathcal{G}'} \inf_{z \in A} d' \circ (f \times f)(x, z) \\ &\leq \sup_{d \in \mathcal{G}} \inf_{z \in A} d(x, z) \\ &= \delta(x, A). \end{aligned}$$

$1 \Rightarrow 5$. Let $x \in f(t_\epsilon(A))$. Then there exists $y \in t_\epsilon(A)$ such that $f(y) = x$. Then we have

$$\delta'(x, f(A)) = \delta'(f(y), f(A)) \leq \delta(y, A) \leq \epsilon.$$

Hence $x \in t'_\epsilon(f(A))$. This proves $f(t_\epsilon(A)) \subset t'_\epsilon(f(A))$.

$5 \Rightarrow 1$. Let $x \in X$ and $A \subset X$. Then we have $x \in A^{(\delta(x,A))}$. We get that $f(x) \in f(A)^{(\delta(x,A))'}$ and hence $\delta'(f(x), f(A)) \leq \delta(x, A)$. $\square$

It should be pointed out that if in the fourth characterization of the foregoing proposition, the gauge $\mathcal{G}'$ is replaced by a basis, then the results remain valid.

Some interesting examples of contractions are given in the following propositions, where the space $\mathbb{P}$, introduced in example 1.5.14, plays the main role.

Proposition 1.6.3 *Let (X, δ) be an approach space. Then, for any $A \in 2^X$ the distance functional*

$$\delta_A : (X, \delta) \to \mathbb{P} : x \mapsto \delta(x, A)$$

is a contraction.

Proof. Let $x \in X$ and $B \subset X$. If $B = \emptyset$, then

$$\delta_{\mathbb{P}}(\delta_A(x), \delta_A(B)) = \delta(x, B) = \infty.$$

Otherwise, since

$$\delta_{\mathbb{P}}(\delta_A(x), \delta_A(B)) = \big(\delta(x, A) - \sup_{b \in B} \delta(b, A)\big) \vee 0$$

the result is an immediate consequence of proposition 1.1.2. $\square$

Proposition 1.6.4 *Let (X, δ) be an approach space. Then, for any filter $\mathcal{F} \in \boldsymbol{F}(X)$ the functions*

$$\lambda\mathcal{F} : (X, \delta) \to \mathbb{P}$$

and

$$\alpha\mathcal{F} : (X, \delta) \to \mathbb{P}$$

are contractions.

Proof. First we prove that $\lambda\mathcal{F}$ is a contraction. If $A = \emptyset$, then we have $\delta_{\mathbb{P}}(\lambda\mathcal{F}(x), \lambda\mathcal{F}(A)) = \infty$ and $\delta(x, A) = \infty$. Suppose now that $A \neq \emptyset$. Then we have

$$\begin{aligned}
\delta_{\mathbb{P}}(\lambda\mathcal{F}(x), \lambda\mathcal{F}(A)) &= \big(\lambda\mathcal{F}(x) - \sup_{a \in A} \lambda\mathcal{F}(a)\big) \vee 0 \\
&= \big(\sup_{F \in \sec(\mathcal{F})} \delta(x, F) - \sup_{a \in A} \sup_{F \in \sec(\mathcal{F})} \delta(a, F)\big) \vee 0 \\
&= \sup_{F \in \sec(\mathcal{F})} \big(\delta(x, F) - \sup_{a \in A} \delta(a, F)\big) \vee 0 \\
&\leq \sup_{F \in \sec(\mathcal{F})} \delta(x, A) \quad \text{by proposition 1.1.2.} \\
&= \delta(x, A).
\end{aligned}$$

The proof for $\alpha\mathcal{F}$ is more or less the same. First consider $A = \emptyset$, then $\delta_{\mathbb{P}}(\alpha\mathcal{F}(x), \alpha\mathcal{F}(A)) = \infty$ and $\delta(x, A) = \infty$. Now suppose that $A \neq \emptyset$. Then we have

$$\begin{aligned}
\delta_{\mathbb{P}}(\alpha\mathcal{F}(x), \alpha\mathcal{F}(A)) &= \big(\alpha\mathcal{F}(x) - \sup_{a\in A} \alpha\mathcal{F}(a)\big) \vee 0 \\
&= \big(\sup_{F\in\mathcal{F}} \delta(x, F) - \sup_{a\in A} \sup_{F\in\mathcal{F}} \delta(a, F)\big) \vee 0 \\
&= \sup_{F\in\mathcal{F}} \big(\delta(x, F) - \sup_{a\in A} \delta(a, F)\big) \vee 0 \\
&\leq \sup_{F\in\mathcal{F}} \delta(x, A) \quad \text{by proposition 1.1.2.} \\
&= \delta(x, A).
\end{aligned}$$

□

1.7 The topological construct AP

Approach spaces form the objects and contractions form the morphisms of a construct that we will denote by **AP**. For the fundamental theory of topological constructs we refer to Adámek *et al.* [1].

Theorem 1.7.1 *Given approach spaces $(X_j, \lambda_j)_{j\in J}$, defined by means of their limit operators, consider the source*

$$(f_j : X \to (X_j, \lambda_j))_{j\in J}$$

*in **AP**. Then the initial limit operator on X is given by*

$$\lambda\mathcal{F} = \sup_{j\in J} \lambda_j(f_j(\mathcal{F})) \circ f_j \qquad \forall \mathcal{F} \in \boldsymbol{F}(X).$$

Proof. First we will prove that λ is indeed a limit operator.
(L1): $\lambda(\operatorname{stack} x)(x) = \sup_{j\in J} \lambda_j(\operatorname{stack} f_j(x))(f_j(x)) = 0$
(L2): Suppose that $\mathcal{F}, \mathcal{G} \in \mathbf{F}(X)$, and $\mathcal{F} \subset \mathcal{G}$. Then we have

$$\lambda\mathcal{G} = \sup_{j\in J} \lambda_j(f_j(\mathcal{G})) \circ f_j \leq \sup_{j\in J} \lambda_j(f_j(\mathcal{F})) \circ f_j = \lambda\mathcal{F}.$$

(L3): Let $(\mathcal{F}_k)_{k\in K}$ be a family of filters on X. Then we have

$$\begin{aligned}
\lambda\Big(\bigcap_{k\in K} \mathcal{F}_k\Big) &= \sup_{j\in J} \lambda_j\Big(f_j\big(\bigcap_{k\in K} \mathcal{F}_k\big)\Big) \circ f_j \\
&= \sup_{j\in J} \lambda_j\Big(\bigcap_{k\in K} f_j(\mathcal{F}_k)\Big) \circ f_j \\
&= \sup_{j\in J} \sup_{k\in K} \lambda_j\big(f_j(\mathcal{F}_k)\big) \circ f_j \\
&= \sup_{k\in K} \sup_{j\in J} \lambda_j\big(f_j(\mathcal{F}_k)\big) \circ f_j \\
&= \sup_{k\in K} \lambda\mathcal{F}_k.
\end{aligned}$$

(L4): Let $\mathcal{F} \in \mathbf{F}(X)$ and let $\mathcal{S} : X \to \mathbf{F}(X)$ be a selection of filters on X. Then we have

$$\begin{aligned}
\lambda\Sigma\mathcal{S}\mathcal{F} &= \sup_{j\in J} \lambda_j\Big(f_j\big(\Sigma\mathcal{S}\mathcal{F}\big)\Big) \circ f_j \\
&= \sup_{j\in J} \lambda_j\Big(\Sigma f_j(\mathcal{S}) f_j(\mathcal{F})\Big) \circ f_j \\
&\leq \sup_{j\in J} \lambda_j\big(f_j(\mathcal{F})\big) \circ f_j + \sup_{j\in J}\sup_{x\in X} \lambda_j f_j(\mathcal{S}(x))(f_j(x)) \\
&= \lambda\mathcal{F} + \sup_{x\in X}\sup_{j\in J} \lambda_j f_j(\mathcal{S}(x))(f_j(x)) \\
&= \lambda\mathcal{F} + \sup_{x\in X} \lambda(\mathcal{S}(x))(x).
\end{aligned}$$

We now prove that all the functions $(f_j)_{j\in J}$ become contractions, when we consider X as an approach space for the limit operator λ. For all $j \in J$ we have

$$\lambda_j(f_j(\mathcal{F})) \circ f_j \leq \sup_{j\in J} \lambda_j(f_j(\mathcal{F})) \circ f_j = \lambda\mathcal{F} \qquad \forall \mathcal{F} \in \mathbf{F}(X).$$

Finally we prove that this limit operator λ is initial. Consider a function $f : (Y, \lambda_Y) \to (X, \lambda)$. We prove that f is a contraction if and only if for all $j \in J, f_j \circ f$ is a contraction. First suppose that f is a contraction. Then we get by using the definition of λ and the fact that f is a contraction, $\forall \mathcal{F} \in \mathbf{F}(X)$:

$$\lambda_j((f_j \circ f)(\mathcal{F})) \circ (f_j \circ f) \leq \lambda f(\mathcal{F}) \circ f \leq \lambda_Y \mathcal{F}.$$

Secondly suppose that for all $j \in J$ $f_j \circ f$ is a contraction. Then we get

$$\lambda(f(\mathcal{F})) \circ f = \sup_{j\in J} \lambda_j(f_j(f(\mathcal{F}))) \circ f_j \circ f \leq \sup_{j\in J} \lambda_Y \mathcal{F} = \lambda_Y \mathcal{F},$$

and hence f is a contraction. □

Theorem 1.7.2 ***AP*** *is a topological construct.*

Proof. **AP** is a construct. The existence of initial structures follows from the previous theorem. For every set X, any class I, any family $(X_i)_{i\in I}$ of approach spaces, and any family $(f_i : X \to X_i)_{i\in I}$ of maps, there exists a unique approach structure λ on X which is initial with respect to the given source. We call this the initial lift of the source. The initial lift is unique. This follows from the fact that for any singleton X, there exists precisely one approach structure on X. □

The following results are immediate consequences of the fact that **AP** is a topological construct.

Corollary 1.7.3 *1. Both initial and final structures exist in* ***AP****.*
2. For any set X, the ***AP****-filter, i.e. the set of all approach structures on X, is a complete lattice.*
3. Constant functions are contractions in ***AP****.*

Proposition 1.7.4 *1. Given a set X, the finest (discrete) approach structure on it is determined by any (and all) of the following structures:*

1. *Distance:* $\delta : X \times 2^X \to [0, \infty]$ *where, for all* $x \in X$ *and* $A \subset X$,

$$\delta(x, A) = \begin{cases} 0 & x \in A, \\ \infty & x \notin A. \end{cases}$$

2. *Limit operator:* $\lambda : \boldsymbol{F}(X) \to [0, \infty]^X$ *where, for all* $x \in X$ *and* $\mathcal{F} \in \boldsymbol{F}(X)$,

$$\lambda\mathcal{F} = \begin{cases} \theta_{\{x\}} & \mathcal{F} = \operatorname{stack} x, \\ \infty & \mathcal{F} \neq \operatorname{stack} x. \end{cases}$$

3. *Gauge:* $\mathcal{G} = pqM^\infty(X)$.

4. *Tower:* $(t_\epsilon)_{\epsilon \in \mathbb{R}^+}$ *where, for all* $\epsilon \in \mathbb{R}^+$ *and* $A \subset X$,

$$t_\epsilon(A) = A.$$

2. Given a set X, the coarsest (indiscrete or trivial) approach structure on it is determined by any (and all) of the following structures:

1. *Distance:* $\delta : X \times 2^X \to [0, \infty]$ *where, for all* $x \in X$ *and* $A \subset X$,

$$\delta(x, A) = \begin{cases} 0 & A \neq \emptyset, \\ \infty & A = \emptyset. \end{cases}$$

2. *Limit operator:* $\lambda : \boldsymbol{F}(X) \to [0, \infty]^X$ *where, for all* $\mathcal{F} \in \boldsymbol{F}(X)$,

$$\lambda\mathcal{F} = 0.$$

3. *Gauge:* $\mathcal{G} = \{0\}$.

4. *Tower:* $(t_\epsilon)_{\epsilon \in \mathbb{R}^+}$ *where, for all* $\epsilon \in \mathbb{R}^+$ *and* $A \subset X$,

$$t_\epsilon(A) = \begin{cases} X & A \neq \emptyset, \\ \emptyset & A = \emptyset. \end{cases}$$

Proof. We prove that the distances given in this theorem, define the discrete and the indiscrete approach structures.

Consider the distance

$$\delta(x, A) = \begin{cases} 0 & x \in A, \\ \infty & x \notin A \end{cases}$$

on X. We have to prove that every function $f : (X, \delta) \to (X', \delta')$ is a contraction. Let $x \in X$ and $A \subset X$. Consider first the possibility that $x \in A$ (and hence $f(x) \in f(A)$). Then we have $\delta'(f(x), f(A)) = 0 = \delta(x, A)$. We now focus on the case where $x \notin A$. Then we have $\delta'(f(x), f(A)) \leq \infty = \delta(x, A)$. So f is a contraction.

Now we consider the distance given by

$$\delta(x, A) = \begin{cases} 0 & A \neq \emptyset, \\ \infty & A = \emptyset \end{cases}$$

on X. We have to prove that every function $f : (X', \delta') \to (X, \delta)$ is a contraction. Let $x \in X$ and $A \subset X$. First consider the case where $A = \emptyset$. Then we get $\delta(f(x), f(A)) = \delta(f(x), \emptyset) = \infty = \delta'(x, A)$. Now we require that $A \neq \emptyset$. Then $f(A) \neq \emptyset$ and hence $\delta(f(x), f(A)) = 0$. Then we have $\delta(f(x), f(A)) = 0 \leq \delta(x, A)$, so f is again a contraction. □

In the following theorem we describe the initial gauge.

Theorem 1.7.5 *Given approach spaces $(X_j, \mathcal{G}_j)_{j \in J}$, defined by means of their gauges, consider the source*

$$(f_j : X \to (X_j, \mathcal{G}_j))_{j \in J}$$

*in **AP**. If, for each $j \in J$, $\mathcal{H}_j$ is a basis for the gauge $\mathcal{G}_j$, then a basis for the initial gauge on X is given by*

$$\mathcal{H} := \Big\{ \sup_{j \in K} d_j \circ (f_j \times f_j) \mid K \in 2^{(J)}, \forall j \in K : d_j \in \mathcal{H}_j \Big\}.$$

Proof. We have to prove that $\mathcal{H}$ is an ideal basis and that $\mathcal{G} = \tilde{\mathcal{H}}$ is the initial gauge on X for the given source.

First we prove that $\mathcal{H}$ is an ideal basis. Consider $d, e \in \mathcal{H}$. This means that there exist subsets $K, K' \in 2^{(J)}$ such that

$$d = \sup_{j \in K} d_j \circ (f_j \times f_j) \qquad \forall j \in K : d_j \in \mathcal{H}_j,$$

and

$$e = \sup_{l \in K'} e_l \circ (f_l \times f_l) \qquad \forall l \in K' : e_l \in \mathcal{H}_l.$$

Then we have

$$d \vee e = \sup_{j \in K} d_j \circ (f_j \times f_j) \vee \sup_{l \in K'} e_l \circ (f_l \times f_l).$$

Every $\mathcal{H}_j$ is an ideal basis, which means that for all $j \in J$ there exists a $c_j \in \mathcal{H}_j$ such that $d_j \vee e_j \leq c_j$, for all $d_j, c_j \in \mathcal{H}_j$. From this we get that

$$d \vee e \leq \sup_{j \in K \cup K'} c_j \circ (f_j \times f_j),$$

which is an element in $\mathcal{H}$.

By definition of $\mathcal{H}$ we get that all functions $f_j : (X, \mathcal{G}) \to (X_j, \mathcal{G}_j), j \in J$ are contractions.

Consider now a function $f : (Y, \mathcal{G}_Y) \to (X, \mathcal{G})$. We prove that f is a contraction if and only if for all $j \in J, f_j \circ f$ is a contraction. First suppose that f is a contraction. For all $j \in J$ we have that f_j is a contraction, hence $f_j \circ f$ is a

contraction for all $j \in J$. Now we suppose that for all $j \in J$ the functions $f_j \circ f$ are contractions. Consider $d' \in \mathcal{G} = \tilde{\mathcal{H}}$. Choose $y \in Y$, $\epsilon > 0$ and $\omega < \infty$. Then we have that there exists a subset $K \subset J$ and for all $j \in K$ there exist $d_j \in \mathcal{H}_j$ such that

$$\begin{aligned} d' \circ (f \times f)(y, \cdot) \wedge \omega &\leq \sup_{j \in K} d_j \circ (f_j \times f_j) \circ (f \times f)(y, \cdot) + \epsilon \\ &= \sup_{j \in K} d_j \circ \big((f_j \circ f) \times (f_j \circ f)\big)(y, \cdot) + \epsilon. \end{aligned}$$

Now we have that $\sup_{j \in K} d_j \circ ((f_j \circ f) \times (f_j \circ f)) \in \mathcal{G}_Y$. Because of the fact that $y \in X$, $\epsilon > 0$ and $\omega < \infty$ were chosen arbitrary, we get that $d' \circ (f \times f) \in \mathcal{G}_Y$. This proves that f is a contraction. □

We end this section by showing that **AP**, as **TOP**, is simply generated, i.e. it has an **initially dense object**.

Theorem 1.7.6 *$\mathbb{P}$ is initially dense in **AP**. More precisely, for any approach space (X, δ), the source*

$$(\delta_A : (X, \delta) \to \mathbb{P})_{A \in 2^X}$$

is initial.

Proof. From proposition 1.6.3 we already know that if δ_{in} stands for the initial distance, then $\delta_{in} \leq \delta$. Conversely, if λ (respectively $\lambda_{in}, \lambda_{\mathbb{P}}$) stands for the limit operator associated with δ (respectively $\delta_{in}, \delta_{\mathbb{P}}$), then it follows from theorem 1.7.1 that, for any $\mathcal{U} \in \mathbf{U}(X)$,

$$\begin{aligned} \lambda_{in}\mathcal{U}(x) &= \sup_{A \in 2^X} \lambda_{\mathbb{P}}\big(\delta_A(\mathcal{U})\big)\big(\delta_A(x)\big) \\ &= \sup_{A \in 2^X} \sup_{U \in \mathcal{U}} \delta_{\mathbb{P}}\big(\delta_A(x), \delta_A(U)\big) \\ &\geq \sup_{U \in \mathcal{U}} \delta_{\mathbb{P}}\big(\delta_U(x), \delta_U(U)\big) \\ &\geq \sup_{U \in \mathcal{U}} \delta_U(x) \\ &= \lambda\mathcal{U}(x), \end{aligned}$$

which proves our claim. □

Chapter 2

TOPOLOGICAL APPROACH SPACES

Topological spaces can be viewed as special types of approach spaces, more precisely, **TOP**, the category of topological spaces and continuous maps, can be embedded as a full and isomorphism-closed subconstruct of **AP**. In this chapter we will describe the embedding. We will prove the pleasant result that **TOP** is very nicely embedded in **AP**, i.e. the category **TOP** can be embedded in **AP** as a simultaneously concretely reflective and concretely coreflective subconstruct.

For this chapter, we refer to the second chapter in the book on approach spaces by R. Lowen [12].

2.1 Topological approach structures

Whenever we say that $(X, \mathcal{T})$ is a topological space, $\mathcal{T}$ will stand for the collection of open sets. Structures derived from $\mathcal{T}$, such as the associated closure operator, will be denoted, for example, by $\mathrm{cl}_{\mathcal{T}}$. If no confusion can arise we may also drop the reference to $\mathcal{T}$.

Given a topological space $(X, \mathcal{T})$, we associate with it a natural approach space in the following way. We define the function

$$\delta_{\mathcal{T}} : X \times 2^X \to [0, \infty]$$

by

$$\delta_{\mathcal{T}}(x, A) := \begin{cases} 0 & \text{if } x \in \mathrm{cl}_{\mathcal{T}}(A), \\ \infty & \text{if } x \notin \mathrm{cl}_{\mathcal{T}}(A). \end{cases}$$

Proposition 2.1.1 *Suppose $(X, \mathcal{T}) \in |\boldsymbol{TOP}|$. Then the function*

$$\delta_{\mathcal{T}} : X \times 2^X \to [0, \infty]$$

is a distance on X.

Proof. We prove that $\delta_{\mathcal{T}}$ is indeed a distance.
(D1): Since $x \in \mathrm{cl}_{\mathcal{T}}(\{x\})$, we immediately have $\delta_{\mathcal{T}}(x, \{x\}) = 0$.

(D2): We know that $\mathrm{cl}_{\mathcal{T}}(\emptyset) = \emptyset$, hence $\delta_{\mathcal{T}}(x, \emptyset) = \infty$.
(D3): For any $A, B \in 2^X$, we have that $\mathrm{cl}_{\mathcal{T}}(A \cup B) = \mathrm{cl}_{\mathcal{T}}(A) \cup \mathrm{cl}_{\mathcal{T}}(B)$. So it follows that $\delta_{\mathcal{T}}(x, A \cup B) = \min(\delta_{\mathcal{T}}(x, A), \delta_{\mathcal{T}}(x, B))$.
(D4): For all $\epsilon < \infty$, we have $A^{(\epsilon)} = \mathrm{cl}_{\mathcal{T}}(A)$ and $A^{(\infty)} = X$. This gives us $\delta_{\mathcal{T}}(x, A) \leq \delta_{\mathcal{T}}\big(x, A^{(\epsilon)}\big) + \epsilon$. □

Definition 2.1.2 *An approach space of type $(X, \delta_{\mathcal{T}})$, for some topology $\mathcal{T}$ on X, will be called a* ***topological approach space****. A distance of type $\delta_{\mathcal{T}}$ will be called a* ***topological distance****.*

This way of looking at topological spaces involves no more than a change of language. Instead of saying that a point is in the closure of a set, we say it has zero distance to the set. Saying that a point is not in the closure of a set becomes saying that the point has an infinite distance to the set.

If $\mathcal{F}$ is a filter on a topological space X, then we denote by $\mathrm{adh}\,\mathcal{F}$ the set of adherence points of $\mathcal{F}$ and by $\lim \mathcal{F}$ the set of limit points of $\mathcal{F}$. If $x \in X$, then we denote the fact that $\mathcal{F}$ adheres to x by $\mathcal{F} \dashv x$ and the fact that $\mathcal{F}$ converges to x by $\mathcal{F} \to x$.

In the sequel we will require the following type of characteristic maps. For any $A \subset X$, we define

$$\theta_A : X \to [0, \infty] : x \mapsto \begin{cases} 0 & x \in A, \\ \infty & x \notin A. \end{cases}$$

We will call this map the **indicator** of A. The set of all such indicators on X will be denoted by $\mathrm{Ind}(X)$.

Proposition 2.1.3 *Suppose $(X, \mathcal{T}) \in |\boldsymbol{TOP}|$. Then the adherence and limit operator in $(X, \delta_{\mathcal{T}})$ are determined by:*

1. $\alpha_{\mathcal{T}}\mathcal{F} = \theta_{\mathrm{adh}\,\mathcal{F}} \qquad \forall \mathcal{F} \in \boldsymbol{F}(X)$,
2. $\lambda_{\mathcal{T}}\mathcal{F} = \theta_{\lim \mathcal{F}} \qquad \forall \mathcal{F} \in \boldsymbol{F}(X)$.

Proof. We give the proof for the adherence operator; the one for the limit operator is precisely the same. It follows from definition 1.5.15 that, for any $\mathcal{F} \in \mathbf{F}(X)$,

$$\begin{aligned} \alpha_{\mathcal{T}}\mathcal{F} &= \sup_{F \in \mathcal{F}} (\delta_{\mathcal{T}})_F \\ &= \sup_{F \in \mathcal{F}} \theta_{\mathrm{cl}_{\mathcal{T}}(F)} \\ &= \theta_{\bigcap_{F \in \mathcal{F}} \mathrm{cl}_{\mathcal{T}}(F)} \\ &= \theta_{\mathrm{adh}\,\mathcal{F}}. \end{aligned}$$

□

The interpretation of the foregoing result is as follows. The distance a point is away from being a limit point of a filter is either zero or infinite, depending on whether it is a topological limit point or not. Notice that this again confirms

the plausibility of our intuitive interpretation of limit and adherence in approach spaces. A limit operator of type $\lambda_{\mathcal{T}}$ will be called a **topological limit operator** and an adherence operator of type $\alpha_{\mathcal{T}}$ will be called a **topological adherence operator**.

Proposition 2.1.4 *Suppose $(X, \mathcal{T}) \in |\boldsymbol{TOP}|$. Then the gauge in $(X, \delta_{\mathcal{T}})$ is given by*

$$\mathcal{G}_T := \{d \in pqM^{\infty}(X) \mid \mathcal{T}_d \subset \mathcal{T}\},$$

where $\mathcal{T}_d$ stands for the topology generated by d.

Proof. From theorem 1.5.3 we obtain that

$$\begin{aligned}
\mathcal{G}_{\mathcal{T}} &= \{d \in pq\mathrm{M}^{\infty}(X) \mid \forall A \subset X, \forall x \in X : \inf_{a \in A} d(x,a) \leq \delta_{\mathcal{T}}(x, A)\} \\
&= \{d \in pq\mathrm{M}^{\infty}(X) \mid \forall A \subset X : x \in \mathrm{cl}_{\mathcal{T}}(A) \Rightarrow \forall a \in A : d(x,a) = 0\} \\
&= \{d \in pq\mathrm{M}^{\infty}(X) \mid \forall A \subset X : x \in \mathrm{cl}_{\mathcal{T}}(A) \Rightarrow x \in \mathrm{cl}_{\mathcal{T}_d}(A)\} \\
&= \{d \in pq\mathrm{M}^{\infty}(X) \mid \mathcal{T}_d \subset \mathcal{T}\}.
\end{aligned}$$

□

A gauge of type $\mathcal{G}_{\mathcal{T}}$ will be called a **topological gauge**.

Proposition 2.1.5 *Suppose that $(X, \mathcal{T}) \in |\boldsymbol{TOP}|$. Then the tower in $(X, \delta_{\mathcal{T}})$ is given by the family $(t_\epsilon^{\mathcal{T}})_{\epsilon \in \mathbb{R}^+}$ where*

$$t_\epsilon^{\mathcal{T}} : 2^X \to 2^X : A \mapsto \mathrm{cl}_{\mathcal{T}}(A) \qquad \forall \epsilon \in \mathbb{R}^+.$$

Proof. For any $\epsilon \in \mathbb{R}^+$ and $A \subset X$, we have

$$\begin{aligned}
t_\epsilon^{\mathcal{T}}(A) &= \{x \in X \mid \delta_{\mathcal{T}}(x, A) \leq \epsilon\} \\
&= \{x \in X \mid \delta_{\mathcal{T}}(x, A) = 0\} \\
&= \mathrm{cl}_{\mathcal{T}}(A).
\end{aligned}$$

□

The pretopological structures of a tower in **TOP** are all equal to the given topological structure. So here too, as was expected, no new structure is involved. A tower of the type $t^{\mathcal{T}}$ will be called a **topological tower**.

The foregoing results show that we can identify topological spaces with certain types of approach spaces. The next result gives an internal characterization of these spaces.

Theorem 2.1.6 *An approach space (X, δ) is topological if and only if*

$$\delta(X \times 2^X) = \{0, \infty\}.$$

Proof. The only-if part follows from the definition of a topological approach space. To show the if part it suffices to note that if, for all $A \subset X$, we put $\mathrm{cl}(A) := \{x \in X \mid \delta(x, A) = 0\}$, then cl is a topological closure operator and δ is the associated distance. □

The result in theorem 2.1.6, in combination with proposition 1.7.4 , shows that, on any set, both the finest and the coarsest approach structures are topological.

2.2 The embedding of TOP in AP

In the foregoing section we have seen that a topological space can be seen as a special type of approach space. The fact that **TOP** is concretely embedded in **AP** is a consequence of the following proposition.

Proposition 2.2.1 *Let $(X, \mathcal{T})$ and $(X', \mathcal{T}')$ be topological spaces and consider the function $f : X \to X'$. Then the following are equivalent:*

1. *$f : (X, \mathcal{T}) \to (X', \mathcal{T}')$ is continuous,*
2. *$f : (X, \delta_{\mathcal{T}}) \to (X', \delta_{\mathcal{T}'})$ is a contraction.*

Proof. If $A \subset X$, then it follows from proposition 2.1.5 that

$$f(\mathrm{cl}_{\mathcal{T}} A) \subset \mathrm{cl}_{\mathcal{T}'}(f(A)) \Leftrightarrow \forall \epsilon \in \mathbb{R}^+ : f\big(A^{(\epsilon)}\big) \subset \big(f(A)\big)^{(\epsilon)'}.$$

This, in combination with theorem 1.6.2 proves our claim. □

Corollary 2.2.2 *The functor*

$$\begin{array}{ccc} \boldsymbol{TOP} & \to & \boldsymbol{AP} \\ (X, \mathcal{T}) & \rightarrowtail & (X, \delta_{\mathcal{T}}) \\ f & \rightarrowtail & f \end{array}$$

*is a full concrete embedding of **TOP** in **AP**.*

We will now prove that **TOP** is actually very nicely embedded in **AP**. In contrast to most known topological constructs which do not have subcontructs which are simultaneously embedded as concretely reflective and concretely coreflective subconstructs, such as for instance **TOP** itself, we will prove that **TOP** is simultaneously concretely reflective and concretely coreflective embedded in **AP**.

We recall that two important aspects of the fact that **TOP** is reflectively embedded in **AP** are: (1) for each approach structure, there exists a finest coarser topological structure on the same underlying set and (2) initial structures of topological spaces are the same whether they are taken in **TOP** or in **AP**.

Proposition 2.2.3 *For any space $(X, \delta) \in |\boldsymbol{AP}|$, the function* $\mathrm{cl} : 2^X \to 2^X$, *defined by*

$$\mathrm{cl}(A) := \{x \in X \mid \delta(x, A) < \infty\} \qquad \forall A \in 2^X,$$

is a pretopological closure operator.

Proof. This follows from (D1), (D2) and (D3). □

We will now give a result about the embedding of **TOP** in **PRTOP**, the topological category of pretopological spaces and continuous maps, which we will need later on.

Proposition 2.2.4 ***TOP*** *is a concretely reflective subcategory of* ***PRTOP****. Given a pretopological space* (X, cl)*, its* ***TOP****-reflection is given by*

$$\mathrm{id}_X : (X, \mathrm{cl}) \to (X, \mathrm{cl}^*),$$

where cl^* *is the closure operator defined by the following transfinite procedure. Given a set* $A \subset X$*, we define*

$$\begin{array}{ll} \mathrm{cl}^0(A) := A, & \\ \mathrm{cl}^{\alpha+1}(A) := \mathrm{cl}(\mathrm{cl}^\alpha(A)) & \alpha \text{ any ordinal,} \\ \mathrm{cl}^\beta(A) := \bigcup_{\alpha<\beta} \mathrm{cl}^\alpha(A) & \beta \text{ a limit ordinal.} \end{array}$$

Then $\mathrm{cl}^*(A) := \mathrm{cl}^\gamma(A)$ *where* γ *is such that* $\mathrm{cl}^{\gamma+1}(A) = \mathrm{cl}^\gamma(A)$*.*

Proof. First we show that $\mathrm{id}_X : (X, \mathrm{cl}) \to (X, \mathrm{cl}^*)$ is continuous. For all $x \in X$ and every $A \subset X$, we have that

$$x \in \mathrm{cl}(A) \Rightarrow x \in \mathrm{cl}^*(A).$$

This means that $\mathrm{cl}(A) \subset \mathrm{cl}^*(A)$ and hence

$$\mathrm{id}_X(\mathrm{cl}(A)) \subset \mathrm{cl}^*(\mathrm{id}_X(A)).$$

This proves that id_X is continuous.

Now suppose that $(Y, \mathrm{cl}_Y) \in |\mathbf{TOP}|$ and that $f : (X, \mathrm{cl}) \to (Y, \mathrm{cl}_Y)$ is a continuous function. We have to prove that $f : (X, \mathrm{cl}^*) \to (Y, \mathrm{cl}_Y)$ is continuous as a function between pretopological spaces. Consider $A \subset X$. We will prove that $f(\mathrm{cl}^*(A)) \subset \mathrm{cl}_Y(f(A))$. We prove this using a transfinite induction.

1. Since $\mathrm{cl}^0(A) = A$, we immediately have $f\big(\mathrm{cl}^0(A)\big) = f(A) \subset \mathrm{cl}_Y(f(A))$.

2. Let α be an ordinal. We take as induction hypothesis

$$f\big(\mathrm{cl}^\alpha(A)\big) \subset \mathrm{cl}_Y(f(A)).$$

   Then we have

$$\begin{array}{rcl} f\big(\mathrm{cl}^{\alpha+1}(A)\big) & = & f\big(\mathrm{cl}(\mathrm{cl}^\alpha(A))\big) \\ & \subset & \mathrm{cl}_Y\big(f(\mathrm{cl}^\alpha(A))\big) \\ & \subset & \mathrm{cl}_Y\big(\mathrm{cl}_Y(f(A))\big) \\ & = & \mathrm{cl}_Y\big(f(A)\big). \end{array}$$

3. Now suppose $x \in \mathrm{cl}^\beta(A)$, for β a limit ordinal. Then there exists an ordinal $\alpha < \beta$, such that $x \in \mathrm{cl}^\alpha(A)$. By making use of the previous step, this gives us $f(x) \in \mathrm{cl}_Y(f(A))$, hence $f\big(\mathrm{cl}^\beta(A)\big) \subset \mathrm{cl}_Y(f(A))$.

□

Theorem 2.2.5 ***TOP*** *is embedded as a concretely reflective subconstruct of* ***AP****. For any space* $(X, \delta) \in |\boldsymbol{AP}|$*, its* ***TOP****-reflection is given by*

$$\mathrm{id}_X : (X, \delta) \to (X, \delta^{tr}),$$

where δ^{tr} *is the distance determined by the topological reflection of the pretopological closure operator* cl*, defined as* $\mathrm{cl}\, A := \{x \in X \mid \delta(x, A) < \infty\}$*.*

Proof. First we verify that $\mathrm{id}_X : (X, \delta) \to (X, \delta^{tr})$ is a contraction. Let $x \in X$ and $A \subset X$. If $\delta(x, A) = \infty$, then we immediately have $\delta^{tr}(x, A) \leq \delta(x, A)$. Suppose now that $\delta(x, A) < \infty$. Then we have, by definition of cl, that $x \in \mathrm{cl}(A) = \{x \in X \mid \delta(x, A) < \infty\}$, and by construction $x \in \mathrm{cl}^{tr}(A)$, where cl^{tr} is the topological reflection of the pretopological closure operator cl, as explained in proposition 2.2.4. This, however, implies that $\delta^{tr}(x, A) = 0$. Hence $\delta^{tr}(x, A) \leq \delta(x, A)$.

Now suppose that $(Y, \mathcal{T}) \in |\mathbf{TOP}|$ and that

$$f : (X, \delta) \to (Y, \delta_{\mathcal{T}})$$

is a contraction. Then, for any $x \in X$ and $A \subset X$, we have

$$\begin{aligned} x \in \mathrm{cl}(A) \quad &\Rightarrow \quad \delta(x, A) < \infty \\ &\Rightarrow \quad \delta_{\mathcal{T}}(f(x), f(A)) < \infty \\ &\Rightarrow \quad \delta_{\mathcal{T}}(f(x), f(A)) = 0 \\ &\Rightarrow \quad f(x) \in \mathrm{cl}_{\mathcal{T}}(f(A)). \end{aligned}$$

This gives us $f(\mathrm{cl}(A)) \subset \mathrm{cl}_{\mathcal{T}}(f(A))$, which means that $f : (X, \mathrm{cl}) \to (Y, \mathrm{cl}_{\mathcal{T}})$ is a continuous function between pretopological spaces. Making use of the reflection explained in proposition 2.2.4, we get that $f : (X, \mathrm{cl}^{tr}) \to (Y, \mathrm{cl}_{\mathcal{T}})$ is continuous as a function between topological spaces. By proposition 2.2.1, this means that $f : (X, \delta^{tr}) \to (Y, \delta_{\mathcal{T}})$ is a contraction. □

Corollary 2.2.6 ***TOP*** *is closed under the formation of limits and initial structures in* ***AP****. In particular, a product in* ***AP*** *of a family of topological approach spaces is a topological approach space and, likewise, a subspace in* ***AP*** *of a topological approach space is a topological approach space.*

Although, as the previous results show, it is important to know from a structural point of view that **TOP** is reflectively embedded in **AP**, we will not often have recourse to considering the **TOP**-reflection of an approach space. It is easily seen that if the distance is finite, then the **TOP**-reflection is indiscrete. This shows that it will usually not be a very interesting topology. The situation becomes totally different for the dual property, concrete coreflectivity. We recall that two important aspects of the fact that **TOP** is coreflectively embedded in **AP** are: (1) for each approach structure there exists a coarsest finer topological structure on the same underlying set and (2) final structures of topological spaces are the same whether they are taken in **TOP** or in **AP**.

Theorem 2.2.7 ***TOP*** *is embedded as a concretely coreflective subconstruct of* ***AP****. For any space $(X, \delta) \in |\boldsymbol{AP}|$, its* ***TOP****-coreflection is given by*

$$\mathrm{id}_X : (X, \delta^{tc}) \to (X, \delta),$$

where δ^{tc} is the distance associated with the topological closure operator given by $\mathrm{cl}_\delta(A) := \{x \in X \mid \delta(x, A) = 0\}$.

Proof. It is easy to see that cl_δ is indeed a topological closure operator and that $\mathrm{id}_X : (X, \delta^{tc}) \to (X, \delta)$ is a contraction.

Now suppose that $(Y, \mathcal{T}) \in |\mathbf{TOP}|$ and that

$$f : (Y, \delta_{\mathcal{T}}) \to (X, \delta)$$

is a contraction. Then, for any $x \in Y$ and $A \subset Y$ such that $x \in \mathrm{cl}_{\mathcal{T}}(A)$, we have

$$\delta(f(x), f(A)) \leq \delta_{\mathcal{T}}(x, A) = 0,$$

which proves that

$$f : (Y, \delta_{\mathcal{T}}) \to (X, \delta^{tc})$$

is also a contraction. □

Corollary 2.2.8 *$\boldsymbol{TOP}$ is closed under the formation of colimits and final structures in $\boldsymbol{AP}$. In particular, a coproduct in $\boldsymbol{AP}$ of a family of topological approach spaces is a topological approach space and, likewise, a quotient in $\boldsymbol{AP}$ of a topological approach space is a topological approach space.*

In the following proposition we will describe the **TOP**-coreflection of an approach space by means of the most important other basic structures. Given an approach space (X, δ), we will denote the topology underlying the topological coreflection by $\mathcal{T}_\delta$.

Proposition 2.2.9 *For any space $(X, \delta) \in |\boldsymbol{AP}|$, the following hold:*

1. *Convergence in $\mathcal{T}_\delta$ is characterized by*

$$\mathcal{F} \to x \Leftrightarrow \lambda\mathcal{F}(x) = 0 \qquad \forall x \in X, \forall \mathcal{F} \in \boldsymbol{F}(X).$$

2. *The neighborhoods in $\mathcal{T}_\delta$ are characterized by*

$$\mathcal{V}(x) = \{V \in 2^X \mid \exists \epsilon > 0, \exists d \in \mathcal{H} : B_d(x, \epsilon) \subset V\} \qquad \forall x \in X,$$

where $\mathcal{H}$ is a basis for the gauge.

Proof. The proof of the first property follows from

$$\begin{aligned}
\mathcal{F} \to x \quad &\Leftrightarrow \quad x \in \bigcap_{A \in \mathrm{sec}(\mathcal{F})} \mathrm{cl}_\delta(A) \\
&\Leftrightarrow \quad \forall A \in \mathrm{sec}(\mathcal{F}) : \delta(x, A) = 0 \\
&\Leftrightarrow \quad \sup_{A \in \mathrm{sec}(\mathcal{F})} \delta(x, A) = 0 \\
&\Leftrightarrow \quad \lambda\mathcal{F}(x) = 0.
\end{aligned}$$

The proof of the second property follows from

$$\begin{aligned}
V \in \mathcal{V}(x) \quad &\Leftrightarrow \quad x \notin \mathrm{cl}_\delta(X \setminus V) \\
&\Leftrightarrow \quad \exists \epsilon > 0 : \sup_{d \in \mathcal{H}} \inf_{y \in X \setminus V} d(x, y) > \epsilon \\
&\Leftrightarrow \quad \exists \epsilon > 0, \exists d \in \mathcal{H} : \inf_{y \in X \setminus V} d(x, y) > \epsilon \\
&\Leftrightarrow \quad \exists \epsilon > 0, \exists d \in \mathcal{H} : B_d(x, \epsilon) \subset V.
\end{aligned}$$

□

Examples 2.2.10

We refer to the two examples which we constructed in 1.5.14. The distance of the first example is $\delta_{\mathbb{E}} : [0,\infty] \times 2^{[0,\infty]} \to [0,\infty]$, where

$$\delta_{\mathbb{E}}(x,A) := \begin{cases} 0 & x = \infty, A \text{ unbounded}, \\ \infty & x = \infty, A \text{ bounded}, \\ \inf_{a \in A \cap \mathbb{R}^+} |x-a| & x < \infty. \end{cases}$$

The topological coreflection of this space is determined by $([0,\infty], \mathcal{T}_{\mathbb{E}})$, where $\mathcal{T}_{\mathbb{E}}$ is the topology of the Alexandroff compactification of $[0,\infty[$, i.e. the usual (Euclidean) topology.

Proof. δ^{tc} is the distance associated with the closure operator $\mathrm{cl}_{\delta_{\mathbb{E}}}$, where $\mathrm{cl}_{\delta_{\mathbb{E}}}(A) = \{x \in [0,\infty] \mid \delta_{\mathbb{E}}(x,A) = 0\}$.
We have that

$$\delta_{\mathbb{E}}(x,A) = 0 \Leftrightarrow \begin{cases} x = \infty \text{ and } A \text{ unbounded}, \\ \text{ or} \\ x < \infty \text{ and } \inf_{a \in A} |x-a| = 0. \end{cases}$$

We now have to prove that the topology defined by the closure operator $\mathrm{cl}_{\delta_{\mathbb{E}}}$ is the same as the topology of the Alexandroff compactification $\mathcal{T}_{\mathbb{E}}$.

$$\delta_{\mathcal{T}_{\mathbb{E}}}(x,A) := \begin{cases} 0 & x \in \mathrm{cl}_{\mathcal{T}_{\mathbb{E}}}(A), \\ \infty & x \notin \mathrm{cl}_{\mathcal{T}_{\mathbb{E}}}(A). \end{cases}$$

If $x < \infty$, we have that $x \in \mathrm{cl}_{\mathcal{T}_{\mathbb{E}}}(A)$ if and only if $\inf_{a \in A} |x-a| = 0$. If $x = \infty$, we have that $x \in \mathrm{cl}_{\mathcal{T}_{\mathbb{E}}}(A)$ if and only if A is unbounded.

We then see that $\delta_{\mathcal{T}_{\mathbb{E}}}$ coincides with δ^{tc}. □

Notice that this distance has a remarkable property. Although the Alexandroff compactification of $[0,\infty[$ is metrizable, it is of course not metrizable by a metric which extends the usual Euclidean metric. The distance $\delta_{\mathbb{E}}$, however, is the unique distance which "distancizes" the topology of the Alexandroff compactification and which extends the usual metric.

Now consider the distance of the second example given by $\delta_{\mathbb{P}} : [0,\infty] \times 2^{[0,\infty]} \to [0,\infty]$, where

$$\delta_{\mathbb{P}}(x,A) := \begin{cases} (x - \sup A) \vee 0 & A \neq \emptyset, \\ \infty & A = \emptyset. \end{cases}$$

The topological coreflection of this space is determined by $([0,\infty], \mathcal{T}_{\mathbb{P}})$, where

$$\mathcal{T}_{\mathbb{P}} := \{]a,\infty] \mid a \in [0,\infty]\} \cup \{[0,\infty]\}.$$

Proof. δ^{tc} is the distance associated with the closure operator $\mathrm{cl}_{\delta_{\mathbb{P}}}$, where $\mathrm{cl}_{\delta_{\mathbb{P}}}(A) = \{x \in [0,\infty] \mid \delta_{\mathbb{P}}(x,A) = 0\}$.
We have that

$$\delta_{\mathbb{P}}(x,A) = 0 \Leftrightarrow A \neq \emptyset \text{ and } x \leq \sup A.$$

We now have to prove that the closure operator $\mathrm{cl}_{\delta_{\mathbb{P}}}$ is the same as the closure operator determined by the topology $\mathcal{T}_{\mathbb{P}}$.

$$\delta_{\mathcal{T}_{\mathbb{P}}}(x, A) := \begin{cases} 0 & x \in \mathrm{cl}_{\mathcal{T}_{\mathbb{P}}}(A), \\ \infty & x \notin \mathrm{cl}_{\mathcal{T}_{\mathbb{P}}}(A). \end{cases}$$

For $x \in [0, \infty], A \subset [0, \infty], A \neq \emptyset$, we have that

$$\begin{aligned} x \in \mathrm{cl}_{\mathcal{T}_{\mathbb{P}}}(A) &\Leftrightarrow \forall V \in \mathcal{V}_{\mathcal{T}_{\mathbb{P}}}(x) : V \cap A \neq \emptyset \\ &\Leftrightarrow \forall a \in [0, x] :]a, \infty] \cap A \neq \emptyset \\ &\Leftrightarrow x \leq \sup A. \end{aligned}$$

We further explain the last step. Suppose that $x \in \mathrm{cl}_{\mathcal{T}_{\mathbb{P}}}(A)$ and $\sup A < x$, then we have that $]\sup A, \infty] \cap A \neq \emptyset$, which is a contradiction. Hence $x \in \mathrm{cl}_{\mathcal{T}_{\mathbb{P}}}(A)$ implies that $x \leq \sup A$. On the other hand, if $x \leq \sup A$, then we have for all $a \leq x$, $]a, \infty] \cap A \neq \emptyset$. If $]a, \infty] \cap A$ would be empty for some $a \leq x$, this would imply that $\sup A < a$ and hence $\sup A < x$, which would be a contradiction. Hence $x \in \mathrm{cl}_{\mathcal{T}_{\mathbb{P}}}(A)$.

We then see that $\delta_{\mathcal{T}_{\mathbb{P}}}$ coincides with δ^{tc}. □

Proposition 2.2.11 *Let (X, δ) be an approach space. Then*

1. *for any $A \subset X$, the distance functional*

$$\delta_A : (X, \mathcal{T}_\delta) \to ([0, \infty], \mathcal{T}_{\mathbb{P}})$$

is a continuous map,

2. *for any $\mathcal{F} \in \boldsymbol{F}(X)$, both*

$$\lambda\mathcal{F} : (X, \mathcal{T}_\delta) \to ([0, \infty], \mathcal{T}_{\mathbb{P}})$$

and

$$\alpha\mathcal{F} : (X, \mathcal{T}_\delta) \to ([0, \infty], \mathcal{T}_{\mathbb{P}})$$

are continuous maps.

Proof. By theorem 2.2.7 and examples 2.2.10 it suffices to show that the functions $\delta_A : (X, \delta) \to ([0, \infty], \delta_{\mathbb{P}})$, $\lambda\mathcal{F} : (X, \delta) \to ([0, \infty], \delta_{\mathbb{P}})$ and $\alpha\mathcal{F} : (X, \delta) \to ([0, \infty], \delta_{\mathbb{P}})$ are contractions. In proposition 1.6.3 we proved that δ_A is a contraction. In proposition 1.6.4 we showed that the functions $\lambda\mathcal{F}$ and $\alpha\mathcal{F}$ are contractions. □

Chapter 3

METRIC APPROACH SPACES

In the previous chapter we introduced a description of topological spaces by means of various equivalent approach structures. These structures, however, did not contain any numerical information. The only numerical values which emerged were 0 and ∞. The situation becomes more interesting when studying metric spaces. In this chapter we will discover the first indications that the "approach setup" allows for a canonical quantification of certain topological concepts.

In this chapter we will be working with the category $pq\mathbf{MET}^{\infty}$ of ∞pq-metric spaces and non-expansive maps and the category $p\mathbf{MET}^{\infty}$ of ∞p-metric spaces and non-expansive maps.

We have seen that **TOP** is simultaneously reflectively and coreflectively embedded in **AP**, and hence for topological approach spaces, it makes no difference whether we perform constructions, such as the making of limits, colimits, initial, and final structures, in **TOP** or in **AP**. However, for ∞pq-metric approach spaces, which we will introduce in the first section of this chapter, it does make a difference whether we make initial structures of ∞pq-metric approach spaces in $pq\mathbf{MET}^{\infty}$ or in **AP**. This will be shown in the third section of this chapter. In the second section of this chapter, we will give a study of convergence in metric approach spaces. In the fourth and last section of this chapter, we will prove that the concretely reflective hull of $pq\mathbf{MET}^{\infty}$ is **AP**.

For this chapter, we refer the reader to the third chapter in the book about approach spaces by R. Lowen [12]. For the section on the concretely reflective hull of $pq\mathbf{MET}^{\infty}$ in **AP**, we refer to an article about intrinsic approach structures on domains by E. Colebunders, S. De Wachter en B. Lowen [7] and to the article about initially dense objects for metrically generated theories by V. Claes [6]

3.1 Metric approach structures

Given an ∞pq-metric space (X, d), we associate with it a natural approach space in the following way.

Proposition 3.1.1 *Suppose* $(X, d) \in |pq\boldsymbol{MET}^\infty|$. *Then*

$$\mathcal{G}_d := \{e \in pqM^\infty(X) \mid e \leq d\}$$

is a gauge on X *and a basis for this gauge is given by* $\mathcal{H} := \{d\}$.

Proof. It is clear that $\mathcal{G}_d$ is an ideal in $pq\mathrm{M}^\infty(X)$. It is also easy to see that $\mathcal{G}_d$ is saturated. Hence $\mathcal{G}_d$ is a gauge. □

Proposition 3.1.2 *Suppose* $(X, d) \in |pq\boldsymbol{MET}^\infty|$. *Then the function*

$$\delta_d : X \times 2^X \to [0, \infty] : (x, A) \mapsto \inf_{a \in A} d(x, a)$$

is the distance on X *associated with the gauge given in proposition 3.1.1.*

Proof. This is merely a special case of theorem 1.5.5 where we take for the gauge basis $\mathcal{H} := \{d\}$. □

Definition 3.1.3 *An approach space of type* (X, δ_d), *for some* ∞pq*-metric* d *on* X, *will be called a* ***metric approach space****. A distance of the type* δ_d *will be called a* ***metric distance*** *and a gauge of the type* $\mathcal{G}_d$ *will be called a* ***metric gauge****.*

We will now describe the other approach structures associated with a metric space. We will see that, unlike for topological spaces, where we basically rediscovered classical topological concepts, for ∞pq-metric spaces we obtain several new concepts.

In what follows, if $(x_n)_n$ is a sequence in X, we will denote by $\langle (x_n)_n \rangle$ the Fréchet filter generated by it. This means that

$$\langle (x_n)_n \rangle := \{F \subset X \mid \exists n : \{x_m \mid m \geq n\} \subset F\}.$$

Proposition 3.1.4 *Suppose* $(X, d) \in |pq\boldsymbol{MET}^\infty|$. *Then the adherence and limit operators in* (X, δ_d) *are determined by:*

1. $\alpha_d \mathcal{F}(x) = \sup_{F \in \mathcal{F}} \inf_{y \in F} d(x, y) \qquad \forall x \in X, \forall \mathcal{F} \in \boldsymbol{F}(X)$,
2. $\lambda_d \mathcal{F}(x) = \inf_{F \in \mathcal{F}} \sup_{y \in F} d(x, y) \qquad \forall x \in X, \forall \mathcal{F} \in \boldsymbol{F}(X)$.

Proof. The first property is an immediate consequence of the definition of the adherence operator given in 1.5.15 and the definition of δ_d. The second property follows from theorem 1.5.12 and proposition 3.1.1. □

Corollary 3.1.5 *Suppose* $(X, d) \in |pq\boldsymbol{MET}^\infty|$. *If* $(x_n)_n$ *is a sequence in* X *and* $x \in X$, *then we have:*

1. $\alpha_d \langle (x_n)_n \rangle (x) = \liminf_{n \to \infty} d(x, x_n)$,
2. $\lambda_d \langle (x_n)_n \rangle (x) = \limsup_{n \to \infty} d(x, x_n)$,

A limit operator of type λ_d will be called a **metric limit operator** and an adherence operator of type α_d will be called a **metric adherence operator**. We will comment on the result of this corollary in the following section.

Proposition 3.1.6 *Suppose $(X, d) \in |pq\boldsymbol{MET}^\infty|$. Then the tower in (X, δ_d) is given by the family $(t^d_\epsilon)_{\epsilon\in\mathbb{R}^+}$, where*

$$t^d_\epsilon : 2^X \to 2^X : A \mapsto \{x \in X \mid \delta_d(x, A) \leq \epsilon\} \qquad \forall \epsilon \in \mathbb{R}^+.$$

Proof. This follows immediately from theorem 1.5.10. □

A tower of type t^d will be called a **metric tower**.

Remarks 3.1.7

Making use of the results of this section it is possible to formulate some of the transitions which we have seen in section 1.5 in a nicer way.

1. *The transition from distances to gauges (1.5.3).* Making use of proposition 3.1.2, if δ is a distance on X, it follows that the associated gauge can be written as
$$\mathcal{G} = \{d \in pq\mathrm{M}^\infty(X) \mid \delta_d \leq \delta\}.$$

2. *The transition from gauges to distances (1.5.5).* Making use of proposition 3.1.2, if $\mathcal{G}$ is a gauge on X, it follows that the associated distance can be written as
$$\delta = \sup_{d\in\mathcal{G}} \delta_d.$$

3. *The transition from gauges to limit operators (1.5.12).* Making use of proposition 3.1.4, if $\mathcal{G}$ is a gauge on X, it follows that the associated limit operator can be written as
$$\lambda = \sup_{d\in\mathcal{G}} \lambda_d.$$

In the sequel we will freely use these new ways of describing these various transitions, but, if necessary, nevertheless refer to the original theorems.

The foregoing results show that we can identify ∞pq-metric spaces with certain types of approach spaces. The following results give internal characterizations of both ∞pq-metric approach spaces and ∞p-metric approach spaces.

Theorem 3.1.8 *For an approach space (X, δ), the following are equivalent:*

1. *(X, δ) is ∞pq-metric,*
2. *for all $x \in X$ and $\mathcal{A} \subset 2^X$, we have $\delta(x, \bigcup \mathcal{A}) = \inf_{A\in\mathcal{A}} \delta(x, A)$,*
3. *for all $x \in X$ and $A \subset X$ we have $\delta(x, A) = \inf_{a\in A} \delta(x, \{a\})$.*

Proof. The implications $1 \Rightarrow 2 \Rightarrow 3$ are evident. To prove $3 \Rightarrow 1$, it suffices to show that

$$d : X \times X \to [0, \infty] : (x, y) \mapsto \delta(x, \{y\})$$

defines an ∞pq-metric on X. Then the third property means precisely that δ is the distance derived from this ∞pq-metric according to proposition 3.1.2.

The fact that d is a ∞pq-metric follows immediately from (D1) and proposition 1.1.2 (4). □

Theorem 3.1.9 *An approach space (X, δ) is ∞p-metric if and only if*

$$\inf_{a \in A} \delta(a, B) = \inf_{b \in B} \delta(b, A) \qquad \forall A, B \in 2^X.$$

Proof. The only-if part is evident. To show the if part, it suffices to notice that symmetry of δ follows from letting A and B be singletons and that, further, for any $a \in X$ and $B \in 2^X$, if we put $A := \{a\}$, then $\delta(a, B) = \inf_{b \in B} \delta(b, \{a\}) = \inf_{b \in B} \delta(a, \{b\})$ and thus, by theorem 3.1.8, δ is a ∞pq-metric. □

3.2 Convergence in metric approach spaces

In this section we will investigate some interesting results concerning convergence in metric approach spaces. In metric approach spaces we are able to work with sequences, which allows us to construct some interesting examples.

Proposition 3.2.1 *Suppose that $(X, d) \in |pq\boldsymbol{MET}^\infty|$. If $\mathcal{F} \in \boldsymbol{F}(X)$ and $x \in X$, then the following hold:*

1. *$\mathcal{F} \dashv x$ in $(X, \mathcal{T}_d)$ if and only if $\alpha_d \mathcal{F}(x) = 0$,*
2. *$\mathcal{F} \to x$ in $(X, \mathcal{T}_d)$ if and only if $\lambda_d \mathcal{F}(x) = 0$.*

Proof. This follows at once from proposition 3.1.4. □

Proposition 3.2.2 *Suppose $(X, d) \in |p\boldsymbol{MET}^\infty|$. If $\mathcal{F} \in \boldsymbol{F}(X)$ and $x \in X$, then the following are equivalent:*

1. *$\mathcal{F} \to x$ in $(X, \mathcal{T}_d)$,*
2. *$\lambda_d \mathcal{F} = d(\cdot, x)$.*

Proof. $1 \Rightarrow 2$. Taking into account the fact that, for any $y \in X$, we have $\delta_d(y, \{x\}) = d(y, x)$, it follows from proposition 1.5.21 that

$$\begin{aligned} \lambda_d \mathcal{F}(y) &\leq \delta_d(y, \{x\}) + \lambda_d \mathcal{F}(x) \\ &= \delta_d(y, \{x\}) \\ &= d(y, x), \end{aligned}$$

which proves one inequality. To prove the other inequality, fix $\epsilon > 0$ and choose for each $F \in \mathcal{F}$ an arbitrary $z_F \in F \cap B(x, \epsilon)$. Then we obtain

$$\begin{aligned} \lambda_d \mathcal{F}(y) + \epsilon &\geq \inf_{F \in \mathcal{F}} d(y, z_F) + d(z_F, x) \\ &\geq d(y, x), \end{aligned}$$

which by arbitrariness of ϵ proves that also $d(y,x) \leq \lambda_d \mathcal{F}(y)$.
$2 \Rightarrow 1$. We have that $\lambda_d \mathcal{F}(x) = d(x,x) = 0$ and from proposition 3.2.1 it follows that $\mathcal{F} \to x$. $\square$

The foregoing result does not hold in $pq\mathbf{MET}^\infty$, as the following example shows.

Example 3.2.3

Consider the real line $\mathbb{R}$ equipped with the ∞pq-metric

$$d(x,y) := (x-y) \vee 0 \qquad \forall x, y \in \mathbb{R}.$$

Fix points $x, y \in \mathbb{R}$ such that $x < y$. The filter $\operatorname{stack} y$ converges to x in $\mathcal{T}_d$. However, from proposition 3.1.4 we have

$$\lambda_d(\operatorname{stack} y)(z) = \inf_{F \in \operatorname{stack} y} \sup_{x \in F} d(z,x) = d(z,y)$$

and $d(z,y) \neq d(z,x)$ whenever $x < z$.

Examples 3.2.4

We now recall the two examples mentioned in the introduction.
1. Consider the real line $\mathbb{R}$ with the usual Euclidean metric d_E, let $\epsilon > 0$, and consider the sequences $(x_n)_n$ and $(y_n)_n$, where

$$x_n := \begin{cases} \epsilon & n \text{ even}, \\ -\epsilon & n \text{ odd} \end{cases}$$

and

$$y_n := \begin{cases} n & n \text{ even}, \\ -n & n \text{ odd}. \end{cases}$$

For the Euclidean topology, neither of these sequences converges. It is clear, however, that from the metric point of view the sequences behave very differently. We will now observe the limit and adherence operators in $(\mathbb{R}, \delta_{d_E})$. From corollary 3.1.5 we obtain for the adherence of the sequence $(x_n)_n$ at any point $x \in \mathbb{R}$,

$$\alpha_{d_E}\langle (x_n)_n \rangle(x) = \liminf_{n \to \infty} |x - x_n| = \begin{cases} |x-\epsilon| & x \geq 0, \\ |x+\epsilon| & x \leq 0, \end{cases}$$

and for the limit of $(x_n)_n$ at any point $x \in \mathbb{R}$,

$$\lambda_{d_E}\langle (x_n)_n \rangle(x) = \limsup_{n \to \infty} |x - x_n| = |x| + \epsilon.$$

The smaller the value of the adherence operator (respectively the limit operator) the closer the point comes to being an adherence point (respectively a limit point). There are two points where $\alpha_{d_E}\langle (x_n)_n \rangle$ attains the minimal value 0, namely $x = \epsilon$ and $x = -\epsilon$. These two points are also the "real" adherence points of the sequence $(x_n)_n$ with respect to the Euclidean topology.

By contrast, $\lambda_{d_E}\langle(x_n)_n\rangle$ nowhere attains the value 0. This accords with the fact that the sequence does not converge with respect to the Euclidean topology. However, $\lambda_{d_E}\langle(x_n)_n\rangle$ does attain a minimal value at the point $x = 0$, which intuitively seems indeed to be the point which best approximates a limit point.

For the sequence $(y_n)_n$ we obtain that

$$\lambda_{d_E}\langle(y_n)_n\rangle(x) = \alpha_{d_E}\langle(y_n)_n\rangle(x) = \infty,$$

for all $x \in \mathbb{R}$. This sequence really does neither adhere nor converge to any point on the real line.

2. Consider the real line $\mathbb{R}$ with the usual Euclidean metric d_E. Let $\varphi : \mathbb{R} \to]-\epsilon, \epsilon[$ be an increasing homeomorphism and let $(r_n)_n$ be an enumeration of the rationals. Consider the sequences $(r_n)_n$ and $(\varphi(r_n))_n$.

From corollary 3.1.5 it follows that, for all $x \in \mathbb{R}$, the adherence of the sequence $(r_n)_n$ is given by

$$\alpha_{d_E}\langle(r_n)_n\rangle(x) = \liminf_{n\to\infty} |r_n - x| = 0.$$

This, according to proposition 3.2.1, is as it should be, since, for the usual topology, the sequence adheres to every point in $\mathbb{R}$.

For the limit of the sequence $(r_n)_n$, also at any point $x \in \mathbb{R}$, we find

$$\lambda_{d_E}\langle(r_n)_n\rangle(x) = \limsup_{n\to\infty} |r_n - x| = \infty,$$

which, in view of the chaotic behavior of the sequence, seems plausible.

For the sequence $(\varphi(r_n))_n$ on the other hand, again from corollary 3.1.5 and for all $x \in \mathbb{R}$, we obtain for the adherence

$$\alpha_{d_E}\langle(\varphi(r_n))_n\rangle(x) = \liminf_{n\to\infty} |\varphi(r_n)-x| = \delta_{d_E}(x,]-\epsilon, \epsilon[) = \begin{cases} |x| - \epsilon & x \leq -\epsilon \text{ or } \epsilon \leq x, \\ 0 & -\epsilon \leq x \leq \epsilon, \end{cases}$$

and for the limit

$$\lambda_{d_E}\langle(\varphi(r_n))_n\rangle(x) = \limsup_{n\to\infty} |\varphi(r_n) - x| = d_E(x, 0) + \epsilon = |x| + \epsilon.$$

These results seem to fit completely with our (metric) intuition. Although, topologically, the behavior of the sequence $(\varphi(r_n))_n$ is the same as that of the sequence $(r_n)_n$, metrically it behaves more nicely. From the point of view of approximation theory, for a sufficiently small ϵ, $(\varphi(r_n))_n$ might be considered to be "almost convergent".

The foregoing examples suggest that in metric approach spaces the behavior of the limit operator is influenced by a sort of "Cauchyness" of the filter. For any set $F \subset X$, we put

$$\operatorname{diam}_d(F) := \sup_{x,y\in F} d(x, y),$$

the d-**diameter** of F. We recall that a filter $\mathcal{F}$ on an ∞pq-metric space (X, d) is a **Cauchy filter** if for every $\epsilon > 0$ there exists a set $F \in \mathcal{F}$ such that the

d-diameter of F is less then ϵ. Given a filter $\mathcal{F}$ on the ∞pq-metric space (X, d), we define the **width** of $\mathcal{F}$ as

$$\omega_d \mathcal{F} := \inf_{F \in \mathcal{F}} \operatorname{diam}_d(F).$$

The width of $\mathcal{F}$ measures if $\mathcal{F}$ is a Cauchy-filter, not with respect to the given space (X, d), but rather with respect to the associated ∞p-metric space (X, d^*), where

$$d^*(x, y) := d(x, y) \vee d(y, x) \qquad \forall x, y \in X.$$

Clearly then $\omega_d \mathcal{F} = 0$ if and only if $\mathcal{F}$ is a Cauchy-filter on (X, d^*).

Proposition 3.2.5 *Suppose $(X, d) \in |pq\boldsymbol{MET}^\infty|$. If $\mathcal{F} \in \boldsymbol{F}(X)$, then we have*

$$\lambda_d \mathcal{F} \leq \alpha_d \mathcal{F} + \omega_d \mathcal{F}.$$

Proof. If $\omega_d \mathcal{F} = \infty$ there is nothing to show. Suppose therefore that $\omega_d \mathcal{F} < \infty$, let $x \in X$, and let $\epsilon > 0$. Choose $F_\epsilon \in \mathcal{F}$ such that

$$\operatorname{diam}_d(F_\epsilon) \leq \omega_d \mathcal{F} + \epsilon.$$

For all $y, z \in F_\epsilon$ we have that

$$d(x, y) \leq d(x, z) + \omega_d \mathcal{F} + \epsilon$$

and thus

$$\sup_{y \in F_\epsilon} d(x, y) \leq \delta_d(x, F_\epsilon) + \omega_d \mathcal{F} + \epsilon,$$

which implies that

$$\begin{aligned} \lambda_d \mathcal{F}(x) &= \inf_{F \in \mathcal{F}} \sup_{y \in F} d(x, y) \\ &\leq \sup_{y \in F_\epsilon} d(x, y) \\ &\leq \delta_d(x, F_\epsilon) + \omega_d \mathcal{F} + \epsilon \\ &\leq \alpha_d \mathcal{F}(x) + \omega_d \mathcal{F} + \epsilon, \end{aligned}$$

which, by arbitrariness of ϵ, proves the result. □

Example 3.2.6

The inequality in the foregoing proposition is best possible, as example 1 of the sequence $(x_n)_n$ in 3.2.4 shows. There we have that $\lambda_{d_E}\langle (x_n)_n \rangle(\epsilon) = \omega_{d_E}\langle (x_n)_n \rangle = 2\epsilon$ and $\alpha_{d_E}\langle (x_n)_n \rangle(\epsilon) = 0$. The fact that $\lambda_{d_E}\langle (x_n)_n \rangle(0) = \alpha_{d_E}\langle (x_n)_n \rangle(0) = \epsilon$, moreover, shows that, in general, the inequality can be strict.

Corollary 3.2.7 *Suppose $(X, d) \in |pq\boldsymbol{MET}^\infty|$. If $\mathcal{F} \in \boldsymbol{F}(X)$ is a Cauchyfilter in (X, d^*), then $\lambda_d \mathcal{F} = \alpha_d \mathcal{F}$.*

Definition 3.2.8 *A filter in a topological space is* ***total*** *if all finer ultrafilters are convergent (not necessarily to the same point).*

A filter $\mathcal{F}$ in a topological space is ***adherent-convergent****, if every open set containing the set of adherence points of $\mathcal{F}$ is a member of $\mathcal{F}$.*

Proposition 3.2.9 *Let $\mathcal{F}$ be a filter basis on a topological space $(X, \mathcal{T})$. Consider the following conditions:*

1. *$\mathcal{F}$ is adherent-convergent and $\operatorname{adh}\mathcal{F}$ is compact.*
2. *$\mathcal{F}$ is total.*

We always have $1 \Rightarrow 2$. If $(X, \mathcal{T})$ is regular, then we have $1 \Leftrightarrow 2$.

Proof. We first show $1. \Rightarrow 2.$ Consider $\mathcal{U} \in \mathbf{U}(X)$ an ultrafilter finer than $\mathcal{F}$. We need to prove that $\mathcal{U}$ is convergent. Therefore we consider $\mathcal{B} = \{U \cap \operatorname{adh}\mathcal{F} \mid U \in \mathcal{U}\}$. For every $U, V \in \mathcal{U}$, we have that $(U \cap \operatorname{adh}\mathcal{F}) \cap (V \cap \operatorname{adh}\mathcal{F}) = (U \cap V) \cap \operatorname{adh}\mathcal{F} \in \mathcal{B}$. It is clear that $\mathcal{B} \neq \emptyset$. We now show that we also have $\emptyset \notin \mathcal{B}$. Suppose that $\emptyset \in \mathcal{B}$. Then there exists $U \in \mathcal{U}$ such that $U \cap \operatorname{adh}\mathcal{F} = \emptyset$. This implies that $\overline{U \cap \operatorname{adh}\mathcal{F}} = \emptyset$ and since $\overline{U} \cap \operatorname{adh}\mathcal{F} \subset \overline{U \cap \operatorname{adh}\mathcal{F}}$, we get that $\overline{U} \cap \operatorname{adh}\mathcal{F} = \emptyset$. Therefore $\operatorname{adh}\mathcal{F} \subset X \setminus \overline{U}$. Since $\mathcal{F}$ is adherent-convergent, this implies that $X \setminus \overline{U} \in \mathcal{F}$. Since $X \setminus \overline{U} \subset X \setminus U$, we eventually get $X \setminus U \in \mathcal{F} \subset \mathcal{U}$. So both $X \setminus U$ and U are in $\mathcal{U}$, which is a contradiction.
Thus $\mathcal{B}$ is a basis for a filter on $\operatorname{adh}\mathcal{F}$ and since $\operatorname{adh}\mathcal{F}$ is compact, there exists $x \in \operatorname{adh}\mathcal{F}$ such that

$$\operatorname{stack}\mathcal{B} \dashv x.$$

This means that for all $V \in \mathcal{V}(x)$ and for all $U \in \mathcal{U}$ we get that $(U \cap \operatorname{adh}\mathcal{F}) \cap V \neq \emptyset$. Hence, for all $V \in \mathcal{V}(x)$ and for all $U \in \mathcal{U}$, $U \cap V \neq \emptyset$. This means that $\mathcal{U} \dashv x$, and because of the fact that $\mathcal{U}$ is an ultrafilter, $\mathcal{U} \to x$.

Suppose now that X is regular, then we prove $2. \Rightarrow 1.$
We first prove that if a filter $\mathcal{F}$ is total, we have that $\mathcal{F}$ is adherent-convergent. Let U be an open neighborhood of $\operatorname{adh}\mathcal{F}$. Suppose that $U \notin \mathcal{F}$. Set $H = X \setminus U$, then we have for all $F \in \mathcal{F}$: $F \cap H \neq \emptyset$, since $F \nsubseteq U$. This means that $\{H\} \cup \mathcal{F}$ is a filterbasis on X. Hence, there exists an ultrafilter $\mathcal{U}$ extending $\{H\} \cup \mathcal{F}$. Then we have $\mathcal{F} \subseteq \mathcal{U}$. $\mathcal{F}$ is total, so we get that $\mathcal{U}$ converges to a certain point x. Then we have $x \in \operatorname{adh}\mathcal{F}$. We get that U is a neighborhood of x disjoint from $H \in \mathcal{U}$. This means that $x \notin \operatorname{adh}\mathcal{U}$, which is a contradiction. So $U \in \mathcal{F}$.
Next we prove that if $\mathcal{F}$ is total, we have that $\operatorname{adh}\mathcal{F}$ is compact. Set $\operatorname{adh}\mathcal{F} = K$. Then we have to prove that every filterbasis $\mathcal{H}$ on K has an adherence point in K. Define $\mathcal{L} = \{U \mid U \text{ is open and contains a member of } \mathcal{H}\}$. Since each open set $U \in \mathcal{L}$ intersects $\operatorname{adh}\mathcal{F}$, $U \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. Thus $\{F \cap U \mid U \in \mathcal{L} \text{ and } F \in \mathcal{F}\}$ is a filterbasis finer than the total filterbasis $\mathcal{F}$. Thus it has an adherence point x. Since X is regular, we can show that $x \in \operatorname{adh}\mathcal{H}$. We know that $\operatorname{adh}\mathcal{H} = \bigcap_{H \in \mathcal{H}} \operatorname{cl} H$. Suppose now that $x \notin \operatorname{adh}\mathcal{H}$. Then there exists $H \in \mathcal{H}$ such that $x \notin \operatorname{cl} H$. Since X is regular, this means that we can find $V \in \mathcal{V}(x)$ and W, an open neighborhood of $\operatorname{cl} H$, such that $V \cap W = \emptyset$. Since $H \subset \operatorname{cl} H \subset W$, we get that $W \in \mathcal{L}$. We also know that $\operatorname{stack}\{U \cap F \mid U \in \mathcal{L}, F \in \mathcal{F}\} \dashv x$, and thus for all $F \in \mathcal{F}$ and for all $U \in \mathcal{L}$ we have that $F \cap U \cap V \neq \emptyset$. If we take $F = X$ and $U = W$, we get $W \cap V \neq \emptyset$, a contradiction. $\square$

Proposition 3.2.10 *Suppose* $(X,d) \in |p\boldsymbol{MET}^{\infty}|$. *If* $\mathcal{F} \in \boldsymbol{F}(X)$ *is total and* $x \in X$, *then the following hold:*

1. $\alpha_d\mathcal{F}(x) = \inf_{y\in \mathrm{adh}_{\mathcal{T}_d}\mathcal{F}} d(x,y) = \delta_d(x, \mathrm{adh}_{\mathcal{T}_d}\mathcal{F})$,
2. $\lambda_d\mathcal{F}(x) = \sup_{y\in \mathrm{adh}_{\mathcal{T}_d}\mathcal{F}} d(x,y)$.

Proof. To prove the first property, suppose that $\delta_d(x, \mathrm{adh}_{\mathcal{T}_d}\mathcal{F}) < \epsilon$. Then there exists $y \in \bigcap_{F\in\mathcal{F}} \mathrm{cl}_{\mathcal{T}_d} F$ for which $d(x,y) < \epsilon$ and it follows that

$$\begin{aligned}
\alpha_d\mathcal{F}(x) &= \sup_{F\in\mathcal{F}} \delta_d(x,F) \\
&= \sup_{F\in\mathcal{F}} \delta_d(x, \mathrm{cl}_{\mathcal{T}_d} F) \\
&\leq d(x,y) \\
&< \epsilon,
\end{aligned}$$

which proves that $\alpha_d\mathcal{F}(x) \leq \delta_d(x, \mathrm{adh}_{\mathcal{T}_d}\mathcal{F})$. To prove the other inequality let $\epsilon > 0$. Since

$$(\mathrm{adh}_{\mathcal{T}_d}\mathcal{F})^{(\epsilon)} \supset \left\{\delta_{\mathrm{adh}_{\mathcal{T}_d}\mathcal{F}} < \epsilon\right\}$$

it follows from proposition 2.2.11 and the fact that $\mathcal{F}$ is adherent-convergent that

$$\delta_d(x, \mathrm{adh}_{\mathcal{T}_d}\mathcal{F}) \leq \delta_d\big(x, (\mathrm{adh}_{\mathcal{T}_d}\mathcal{F})^{(\epsilon)}\big) + \epsilon \leq \alpha_d\mathcal{F}(x) + \epsilon.$$

For the proof of the second property, since an ultrafilter finer than $\mathcal{F}$ converges to a point in $\mathrm{adh}_{\mathcal{T}_d}\mathcal{F}$, it follows from the first property, proposition 1.5.17, and proposition 1.5.18 that

$$\begin{aligned}
\lambda_d\mathcal{F}(x) &= \sup_{\mathcal{U}\in\mathbf{U}(\mathcal{F})} \lambda_d\mathcal{U}(x) \\
&= \sup_{\mathcal{U}\in\mathbf{U}(\mathcal{F})} \delta_d(x, \mathrm{adh}_{\mathcal{T}_d}\mathcal{U}) \\
&\leq \sup_{y\in \mathrm{adh}_{\mathcal{T}_d}\mathcal{F}} d(x,y).
\end{aligned}$$

For the converse inequality it suffices to notice that, for any $y \in \mathrm{adh}_{\mathcal{T}_d}\mathcal{F}$, we have

$$\begin{aligned}
\lambda_d\mathcal{F}(x) &= \inf_{F\in\mathcal{F}} \sup_{z\in F} d(x,z) \\
&= \inf_{F\in\mathcal{F}} \sup_{z\in \mathrm{cl}_{\mathcal{T}_d}(F)} d(x,z) \\
&\geq \inf_{F\in\mathcal{F}} d(x,y) \\
&= d(x,y),
\end{aligned}$$

which proves that $\sup_{y\in \mathrm{adh}_{\mathcal{T}_d}\mathcal{F}} d(x,y) \leq \lambda_d\mathcal{F}(x)$. □

Again, it should be pointed out that the foregoing result does not hold in $pq\mathbf{MET}^{\infty}$.

Example 3.2.11

Consider the set of real numbers $[0, \infty[$ equipped with the ∞pq-metric d_P

$$d_P(x, y) := (x - y) \vee 0 \qquad \forall x, y \in [0, \infty[.$$

Fix a point $x_0 \in]0, \infty[$. For the filter $\mathcal{F} := \operatorname{stack} x_0$ we have $\operatorname{adh}_{\mathcal{T}_{d_P}} \mathcal{F} = [0, x_0]$. Consequently it follows that, for all $x \in [0, \infty[$,

$$\sup_{y \in \operatorname{adh}_{\mathcal{T}_{d_P}} \mathcal{F}} d_P(x, y) = d_P(x, 0)$$

whereas

$$\lambda_{d_P} \mathcal{F}(x) = d_P(x, x_0).$$

Totality too is a necessary condition in proposition 3.2.10.

Example 3.2.12

Consider $\mathbb{R} \setminus \{0\}$ equipped with the usual metric d_E and the usual topology $\mathcal{T}_E$, and consider the sequence $(z_n)_{n \geq 1}$, where

$$z_n := \begin{cases} n & n \text{ even}, \\ \frac{1}{n} & n \text{ odd}. \end{cases}$$

Denote by $\mathcal{F}$ the filter generated by $(z_n)_{n \geq 1}$. Then the adherence and limit of $\mathcal{F}$ are given respectively by

$$\alpha_{d_E} \mathcal{F}(x) = |x| \qquad \forall x \in \mathbb{R} \setminus \{0\}$$

and

$$\lambda_{d_E} \mathcal{F}(x) = \infty \qquad \forall x \in \mathbb{R} \setminus \{0\}.$$

Now clearly, $\mathcal{F}$ is not total and, actually $\operatorname{adh}_{\mathcal{T}_E} \mathcal{F} = \emptyset$. Consequently, for any $x \in \mathbb{R} \setminus \{0\}$, the formulas in proposition 3.2.10 give us $\inf_{y \in \operatorname{adh}_{\mathcal{T}_E} \mathcal{F}} d(x, y) = \infty$, which differs from $\alpha_{d_E} \mathcal{F}(x)$ and $\sup_{y \in \operatorname{adh}_{\mathcal{T}_E} \mathcal{F}} d(x, y) = 0$, which differs from $\lambda_{d_E} \mathcal{F}(x)$.

3.3 The embedding of $pq\mathbf{MET}^\infty$ in AP

In section 3.1 we have seen that ∞pq-metric spaces can be viewed as special types of approach space. That $pq\mathbf{MET}^\infty$ is concretely embedded in **AP** is a consequence of the following proposition.

Proposition 3.3.1 *Let (X, d) and (X', d') be ∞pq-metric spaces and consider the function $f : X \to X'$. Then the following are equivalent:*

1. *$f : (X, d) \to (X', d')$ is non-expansive,*
2. *$f : (X, \delta_d) \to (X', \delta_{d'})$ is a contraction.*

Proof. 1. $\Rightarrow$ 2.:

$$\delta_{d'}(f(x), f(A)) = \inf_{a \in A} d'(f(x), f(a)) \leq \inf_{a \in A} d(x, a) = \delta_d(x, a).$$

2. $\Rightarrow$ 1.:

$$d'(f(x), f(a)) = \delta_{d'}(f(x), \{f(a)\}) \leq \delta_d(x, \{a\}) = d(x, a).$$

□

Corollary 3.3.2 *The functor*

$$\begin{array}{ccc} pq\boldsymbol{MET}^\infty & \to & \boldsymbol{AP} \\ (X, d) & \rightarrowtail & (X, \delta_d) \\ f & \rightarrowtail & f \end{array}$$

is a full concrete embedding of $pq\boldsymbol{MET}^\infty$ *in* $\boldsymbol{AP}$.

In the second chapter we were able to show both concrete reflectivity and concrete coreflectivity of the embedding of **TOP** in **AP**. For $p\mathbf{MET}^\infty$ and $pq\mathbf{MET}^\infty$ we are only able to show concrete coreflectivity of the embedding. As we will see later, it is precisely the fact that neither $pq\mathbf{MET}^\infty$ nor $p\mathbf{MET}^\infty$ is embedded reflectively in **AP** which makes the theory of approach spaces especially interesting.

Theorem 3.3.3 *The construct* $pq\boldsymbol{MET}^\infty$ *is embedded as a concretely coreflective subconstruct of* $\boldsymbol{AP}$. *For any space* $(X, \delta) \in |\boldsymbol{AP}|$, *its* $pq\boldsymbol{MET}^\infty$*-coreflection is given by*

$$\mathrm{id}_X : (X, \delta^{qm}) \to (X, \delta),$$

where δ^{qm} *is the distance determined by the* ∞pq*-metric*

$$d_\delta : X \times X \to [0, \infty] : (x, y) \mapsto \delta(x, \{y\}).$$

Proof. To show that $\mathrm{id}_X : (X, \delta^{qm}) \to (X, \delta)$ is a contraction, let $x \in X$ and let $A \subset X$. Then we have

$$\delta(x, A) \leq \inf_{a \in A} \delta(x, \{a\}) = \inf_{a \in A} d_\delta(x, a) = \delta^{qm}(x, A).$$

Now suppose that $(Y, d) \in |pq\mathbf{MET}^\infty|$ and that

$$f : (Y, \delta_d) \to (X, \delta)$$

is a contraction. Then, for any $x \in Y$ and $A \subset Y$, we have

$$\begin{aligned} \delta^{qm}(f(x), f(A)) &= \inf_{a \in A} \delta(f(x), \{f(a)\}) \\ &\leq \inf_{a \in A} \delta_d(x, \{a\}) \\ &= \delta_d(x, A), \end{aligned}$$

which proves that

$$f : (Y, \delta_d) \to (X, \delta^{qm})$$

is a contraction. □

Corollary 3.3.4 *$pq\boldsymbol{MET}^\infty$ is closed under the formation of colimits and final structures in $\boldsymbol{AP}$. In particular, a coproduct in $\boldsymbol{AP}$ of a family of ∞pq-metric approach spaces is an ∞pq-metric approach space and, likewise, a quotient in $\boldsymbol{AP}$ of an ∞pq-metric approach space is an ∞pq-metric approach space.*

In the following result we describe the $pq\mathbf{MET}^\infty$-coreflection of an approach space by means of the associated gauge. This result is the counterpart of proposition 2.2.9. Note that the ∞pq-metric generating the $pq\mathbf{MET}^\infty$-coreflection of an approach space (X, δ) is denoted by d_δ.

Proposition 3.3.5 *Let $(X, \delta) \in |\boldsymbol{AP}|$. If $\mathcal{H}$ is a basis for the gauge associated with δ then*

$$d_\delta(x, y) = \sup_{d \in \mathcal{H}} d(x, y) \qquad \forall x, y \in X.$$

Proof. This property follows at once from theorem 1.5.5 and theorem 3.3.3. □

Theorem 3.3.6 *The construct $p\boldsymbol{MET}^\infty$ is embedded as a concretely coreflective subconstruct of $\boldsymbol{AP}$. For any space $(X, \delta) \in |\boldsymbol{AP}|$, its $p\boldsymbol{MET}^\infty$-coreflection is given by*

$$\mathrm{id}_X : (X, \delta^m) \to (X, \delta),$$

where δ^m is the distance determined by the ∞p-metric

$$d^*_\delta : X \times X \to [0, \infty] : (x, y) \mapsto \delta(x, \{y\}) \vee \delta(y, \{x\}).$$

Proof. This proof is precisely the same as the one from theorem 3.3.3. □

Corollary 3.3.7 *$p\boldsymbol{MET}^\infty$ is closed under the formation of colimits and final structures in $\boldsymbol{AP}$. In particular, a coproduct in $\boldsymbol{AP}$ of a family of ∞p-metric approach spaces is an ∞p-metric approach space and, likewise, a quotient in $\boldsymbol{AP}$ of an ∞p-metric approach space is an ∞p-metric approach space.*

From theorem 3.3.3 and theorem 3.3.6 it follows that $d^*_\delta = (d_\delta)^*$ where, for any $d \in pq\mathrm{M}^\infty(X)$, d^* stands for the symmetrization of d, given by $d^*(x, y) := d(x, y) \vee d(y, x)$, for all $x, y \in X$. The description of the $p\mathbf{MET}^\infty$-coreflection of an approach space by means of a basis for the gauge is therefore easily deduced from proposition 3.3.5 and theorem 3.3.6. For instance, if $\mathcal{H}$ is a basis for the gauge associated with δ, then $d^*_\delta(x, y) = \sup_{d \in \mathcal{H}} d^*(x, y)$, for all $x, y \in X$.

Examples 3.3.8

Again we refer to the examples which were considered in 1.5.14. The distance of the first example is $\delta_{\mathbb{E}} : [0, \infty] \times 2^{[0,\infty]} \to [0, \infty]$, where

$$\delta_{\mathbb{E}}(x, A) := \begin{cases} 0 & x = \infty, A \text{ unbounded}, \\ \infty & x = \infty, A \text{ bounded}, \\ \inf_{a \in A \cap \mathbb{R}^+} |x - a| & x < \infty. \end{cases}$$

Both the $pq\mathbf{MET}^\infty$-coreflection and the $p\mathbf{MET}^\infty$-coreflection of this approach space are given by $([0,\infty], d_{\mathbb{E}})$ where $d_{\mathbb{E}}$ is the Euclidean metric on $[0,\infty]$, i.e.

$$d_{\mathbb{E}}(x,y) := |x-y| \qquad \forall x,y \in [0,\infty].$$

The distance of the second example is $\delta_{\mathbb{P}} : [0,\infty] \times 2^{[0,\infty]} \to [0,\infty]$, where

$$\delta_{\mathbb{P}}(x,A) := \begin{cases} (x - \sup A) \vee 0 & A \neq \emptyset, \\ \infty & A = \emptyset. \end{cases}$$

The $pq\mathbf{MET}^\infty$-coreflection of this approach space is $([0,\infty], d_{\mathbb{P}})$, where

$$d_{\mathbb{P}}(x,y) := (x-y) \vee 0 \qquad \forall x,y \in [0,\infty]$$

and the $p\mathbf{MET}^\infty$-coreflection is given by $([0,\infty], d_{\mathbb{E}})$.

Proposition 3.3.9 *Let (X,δ) be an approach space. Then*

1. *for any $A \in 2^X$, the distance functional*

$$\delta_A : (X, d_\delta) \to ([0,\infty], d_{\mathbb{P}})$$

is a non-expansive map.

2. *for any $\mathcal{F} \in \boldsymbol{F}(X)$, both*

$$\lambda\mathcal{F} : (X, d_\delta) \to ([0,\infty], d_{\mathbb{P}})$$

and

$$\alpha\mathcal{F} : (X, d_\delta) \to ([0,\infty], d_{\mathbb{P}})$$

are non-expansive maps.

Proof. This follows at once from proposition 1.6.3, proposition 1.6.4, theorem 3.3.3 and example 3.3.8. □

In order to illustrate further the **TOP**-coreflection, the $pq\mathbf{MET}^\infty$-coreflection, and the $p\mathbf{MET}^\infty$-coreflection of an approach space, we will now describe these respectively for a topological, an ∞pq-metric, and an ∞p-metric approach space.

Proposition 3.3.10 *If $(X,\delta_{\mathcal{T}})$ is a topological approach space, then:*

1. *the $pq\boldsymbol{MET}^\infty$-coreflection is given by (X,δ_{d_1}), where d_1 is the ∞pq-metric*

$$d_1(x,y) := \begin{cases} 0 & x \in \mathrm{cl}_{\mathcal{T}}\{y\}, \\ \infty & x \notin \mathrm{cl}_{\mathcal{T}}\{y\}, \end{cases}$$

and

2. *the $p\boldsymbol{MET}^\infty$-coreflection is given by (X,δ_{d_0}), where d_0 is the ∞p-metric*

$$d_0(x,y) := \begin{cases} 0 & x \in \mathrm{cl}_{\mathcal{T}}\{y\} \text{ and } y \in \mathrm{cl}_{\mathcal{T}}\{x\}, \\ \infty & x \notin \mathrm{cl}_{\mathcal{T}}\{y\} \text{ or } y \notin \mathrm{cl}_{\mathcal{T}}\{x\}. \end{cases}$$

Proof. This follows from the definition of $\delta_{\mathcal{T}}$, from theorem 3.3.3 and from theorem 3.3.6. $\square$

The foregoing result has a rather interesting interpretation. In view of the fact that $x \notin \mathrm{cl}_{\mathcal{T}}(\{y\})$ if and only if there exists a neighborhood V of x such that $y \notin V$, it follows that the $pq\mathbf{MET}^{\infty}$-coreflection indicates whether the topological space is T_1 or not. $(X, \mathcal{T})$ is T_1 if and only if d_1 is an ∞-metric. In the same way the $p\mathbf{MET}^{\infty}$-coreflection indicates whether the topological space is T_0 or not. More precisely, $(X, \mathcal{T})$ is T_0 if and only if d_0 is an ∞-metric.

Proposition 3.3.11 *If (X, δ_d) is an ∞pq-metric approach space, then*

1. *the **TOP**-coreflection of (X, δ_d) is given by $(X, \delta_{\mathcal{T}_d})$, where $\mathcal{T}_d$ is the topology generated by d, and*
2. *the p**MET**$^{\infty}$-coreflection of (X, δ_d) is given by (X, δ_{d^*}), where d^* is the ∞p-metric*
$$d^*(x, y) := d(x, y) \vee d(y, x) \qquad \forall x, y \in X.$$

Proof. The first property follows from the second property in proposition 2.2.9. This result indeed implies that the **TOP**-coreflection of (X, δ_d) has as neighborhood system

$$\mathcal{V}(x) = \{V \in 2^X \mid \exists \epsilon > 0 : B_d(x, \epsilon) \subset V\} \qquad \forall x \in X.$$

The second property follows from theorem 3.3.6. $\square$

Proposition 3.3.12 *If (X, δ_d) is an ∞p-metric approach space, then the **TOP**-coreflection of (X, δ_d) is given by $(X, \delta_{\mathcal{T}_d})$, where $\mathcal{T}_d$ stands for the topology generated by d.*

Proof. This is exactly the same as the proof of proposition 3.3.11. $\square$

A fundamental relationship among the different types of structures which we are considering in this work is that of a topology generated by a metric. It is the failure of this relationship to be well behaved with respect to products which is one of the main motivations for considering approach spaces.

What the foregoing results tell us is that this relationship is recaptured in **AP** as a canonical functor, namely the **TOP**-coreflector restricted to $pq\mathbf{MET}^{\infty}$. In the case of an ∞pq-metric space the **TOP**-coreflector gives us the underlying topological space. It is therefore natural to extend this interpretation to the whole of **AP**. Given an arbitrary approach space (X, δ), we will speak of (X, δ^{tc}) (the **TOP**-coreflection of (X, δ)) as the **underlying topological approach space** and of the topology generating (X, δ^{tc}) as the **topology underlying** δ or the topology generated by δ. This topology will be denoted by $\mathcal{T}_\delta$. The situation is clarified in the following diagram.

$$\begin{tikzcd}
pq\mathbf{MET}^{\infty} \arrow[r, "F_1"] \arrow[d, "E"'] & \mathbf{TOP} \\
\mathbf{AP} \arrow[ur, "F_2"'] &
\end{tikzcd}$$

The functor E in this diagram is the embedding of $pq\mathbf{MET}^\infty$ in $\mathbf{AP}$, F_1 is the functor associating with each ∞pq-metric space its underlying topological space, and F_2 is the $\mathbf{TOP}$-coreflector. The diagram commutes and F_2 is an extension of F_1.

It is a fundamental aspect of the theory of approach spaces that $pq\mathbf{MET}^\infty$ and, especially, $p\mathbf{MET}^\infty$ are not epireflectively embedded in $\mathbf{AP}$. In particular neither subconstruct is stable for the formation of products in $\mathbf{AP}$. As the following result shows, this is the only reason why neither $pq\mathbf{MET}^\infty$ nor $p\mathbf{MET}^\infty$ are epireflectively embedded in $\mathbf{AP}$.

Proposition 3.3.13 *$pq\boldsymbol{MET}^\infty$ and $p\boldsymbol{MET}^\infty$ are closed under the formation of subspaces in $\boldsymbol{AP}$.*

Proof. This follows at once from the definitions. □

In the following proposition we give some remarkable results about approach spaces which are at the same time topological and metric. The subconstructs to which they give rise are isomorphic to some well-known subconstructs of $\mathbf{TOP}$. We denote the full subconstruct of $\mathbf{TOP}$ with objects all discrete spaces by $\mathbf{DIS}$, the subconstruct with objects all coproducts of indiscrete spaces by $\mathbf{CIN}$, and the subconstruct with objects all finitely generated spaces by $\mathbf{FING}$. A topolgical space is finitely generated if a point is in the closure of a set A if and only if it is in the closure of one of the points of A.

Proposition 3.3.14 *The following properties hold:*

1. $\boldsymbol{TOP} \cap pq\boldsymbol{MET}^\infty = \boldsymbol{FING}$.
2. $\boldsymbol{TOP} \cap p\boldsymbol{MET}^\infty = \boldsymbol{CIN}$.
3. $\boldsymbol{TOP} \cap \boldsymbol{MET}^\infty = \boldsymbol{DIS}$.

Proof.

1. If $(X, \mathcal{T})$ is finitely generated, then

$$\delta_{\mathcal{T}}(x, A) = \begin{cases} 0 & \text{if there exists an } a \in A \text{ such that } x \in \mathrm{cl}_{\mathcal{T}}\{a\}, \\ \infty & \text{if for all } a \in A : x \notin \mathrm{cl}_{\mathcal{T}}\{a\}. \end{cases}$$

If $\delta_{\mathcal{T}}(x, A) = 0$, then there exists an $a \in A$ such that $\delta_{\mathcal{T}}(x, \{a\}) = 0$ and we get $\delta_{\mathcal{T}}(x, A) = \inf_{a \in A} \delta_{\mathcal{T}}(x, \{a\})$. If $\delta_{\mathcal{T}}(x, A) = \infty$, then we have that for all $a \in A, \delta_{\mathcal{T}}(x, \{a\}) = \infty$, and we get $\delta_{\mathcal{T}}(x, A) = \inf_{a \in A} \delta(x, \{a\})$. This gives us that for every $x \in X$ and $A \subset X$, $\delta_{\mathcal{T}}(x, A) = \inf_{a \in A} \delta_{\mathcal{T}}(x, \{a\})$. By theorem 3.1.9. this means that $(X, \mathcal{T})$ is ∞pq-metric.

Conversely, if (X, δ) is at the same time topological and ∞pq-metric, then we have for any $x \in X$ and $A \subset X$

$$\begin{aligned} x \in \mathrm{cl}_\delta(A) &\Leftrightarrow \delta(x, A) = 0 \\ &\Leftrightarrow \inf_{a \in A} \delta(x, \{a\}) = 0 \\ &\Leftrightarrow \exists a \in A : \delta(x, \{a\}) = 0 \\ &\Leftrightarrow \exists a \in A : x \in \mathrm{cl}_\delta\{a\}. \end{aligned}$$

Hence, the underlying topology is finitely generated.

2. First we notice that if $(X, \mathcal{T})$ is a topological space, we have that all open sets are closed if and only if X is a coproduct of indiscrete spaces.
   If $(X, \mathcal{T})$ is a coproduct of indiscrete spaces, then it follows from the first property that $\delta_{\mathcal{T}}$ is at the same time topological and ∞pq-metric. Suppose that $\delta_{\mathcal{T}} = \delta_d$ for some ∞pq-metric d. Then, if X is the coproduct of spaces $(X_j)_{j \in J}$, it follows that, for any $x, y \in X, d(x, y) = 0$ if and only if there exists $j \in J$ such that $x, y \in X_j$. Hence d is symmetric.
   Conversely, if (X, δ) is at the same time topological and ∞p-metric, then we claim that the open sets and the closed sets in the underlying topology coincide. Let A be an open set, then we prove that A is closed. Let d be an ∞p-metric such that $\delta = \delta_d$. Then we have

$$\begin{aligned} x \in \mathrm{cl}(A) &\Rightarrow \delta(x, A) = 0 \\ &\Rightarrow \inf_{a \in A} d(x, a) = 0 \\ &\Rightarrow \exists \epsilon > 0 : x \in B_d(a, \epsilon) \subset A. \end{aligned}$$

   Hence we get that this topology is a coproduct of indiscrete topologies.

3. If $(X, \mathcal{T})$ is discrete, then it follows from the second property that $\delta_{\mathcal{T}}$ is at the same time topological and ∞p-metric. The discreteness, however, implies that d is separated.
   Conversely, if (X, δ) is at the same time topological and ∞-metric, then we claim that every subset of X coincides with its closure. Let A be a subset of X. We prove that $\mathrm{cl}\, A = A$. Let again d be an ∞-metric, such that $\delta = \delta_d$.

$$\begin{aligned} x \in \mathrm{cl}(A) &\Rightarrow \delta(x, A) = 0 \\ &\Rightarrow \inf_{a \in A} d(x, a) = 0 \\ &\Rightarrow \exists a \in A : d(x, a) = 0 \\ &\Rightarrow \exists a \in A : x = a \\ &\Rightarrow x \in A. \end{aligned}$$

   This proves that this topology is discrete.

□

3.4 The concretely reflective hull of $pq\mathbf{MET}^{\infty}$

At the topological level we know that **TOP** consists precisely of subspaces of products (in **TOP**) of ∞pq-metrizable topological spaces. We will show that this repeats itself at the approach level.

First of all we define

$$d_{\mathbb{P}}^{-1} : [0, \infty] \times [0, \infty] \to [0, \infty] : (x, y) \mapsto (y - x) \vee 0,$$

and for every $\alpha \in [0, \infty[$, we define

$$T_\alpha : ([0, \infty], d_{\mathbb{P}}^{-1}) \to ([0, \infty], d_{\mathbb{P}}) : u \mapsto (\alpha - u) \vee 0.$$

Proposition 3.4.1 *The following assertions hold:*

1. $d_{\mathbb{P}}^{-1}$ *is initially dense in* $\boldsymbol{AP}$.

2. *Consider the source*

$$\big(T_\alpha : ([0,\infty], d_{\mathbb{P}}^{-1}) \to ([0,\infty], d_{\mathbb{P}})\big)_{\alpha\in[0,\infty[}.$$

Then we have
(i) $d_{\mathbb{P}} \circ T_\alpha \leq d_{\mathbb{P}}^{-1}$,
(ii) $d_{\mathbb{P}}^{-1}(x,\cdot) \wedge \omega \leq d_{\mathbb{P}} \circ T_\alpha \times T_\alpha(x,\cdot)$ *if* $\alpha = x + \omega$.

Proof.

1. We first prove that $d_{\mathbb{P}}^{-1}$ is initially dense in **AP**.
Consider the source

$$\big(g_\beta : ([0,\infty], \delta_{\mathbb{P}}) \to ([0,\infty], d_{\mathbb{P}}^{-1})\big)_{\beta\in[0,\infty]},$$

with $g_\beta(y) = (\beta - y) \vee 0$ for $y \in [0,\infty]$. We claim that this source is initial in **AP**. Let

$$\{d_\beta \mid \beta \in [0,\infty]\}$$

be the gauge basis for $([0,\infty], \delta_{\mathbb{P}})$, where $d_\beta(x,y) = (x \wedge \beta - y \wedge \beta) \vee 0$. We prove that for $\beta \in [0,\infty[$ and $y, z \in [0,\infty]$,

$$d_{\mathbb{P}}^{-1} \circ g_\beta \times g_\beta(y,z) = d_\beta(y,z).$$

Writing the explicit forms of both sides we get, for the left hand side

$$\begin{aligned} d_{\mathbb{P}}^{-1} \circ g_\beta \times g_\beta(y,z) &= \big(g_\beta(z) - g_\beta(y)\big) \vee 0 \\ &= \big((\beta - z) \vee 0 - (\beta - y) \vee 0\big) \vee 0 \end{aligned}$$

and for the right hand side

$$d_\beta(y,z) = (y \wedge \beta - z \wedge \beta) \vee 0.$$

Observe that in each of the cases $y \leq z$ or $\beta < z < y$ both sides are zero. In case ($z < y$ and $z \leq \beta \leq y$) both sides are equal $y - z$. These observations remain valid when y and z are ∞. It follows that the approach gauges generated by $\{d_{\mathbb{P}}^{-1} \circ g_\beta \times g_\beta \mid \beta \in [0,\infty[\}$ and $\{d_\beta \mid \beta \in [0,\infty[\}$ coincide. This first collection generates the initial approach structure determined by the given source, the second collection generates $\delta_{\mathbb{P}}$. So we can conclude that $\delta_{\mathbb{P}}$ is the initial approach structure. Using the fact that $\delta_{\mathbb{P}}$ is initially dense in **AP**, we immediately get that $d_{\mathbb{P}}^{-1}$ is initially dense in **AP** by transitivity.

2. (i) For all $y, z \in [0,\infty]$, we have

$$\begin{aligned} d_{\mathbb{P}} \circ T_\alpha \times T_\alpha(y,z) &= d_{\mathbb{P}}\big(T_\alpha(y), T_\alpha(z)\big) \\ &= d_{\mathbb{P}}\big((\alpha - y) \vee 0, (\alpha - z) \vee 0\big) \\ &= \big((\alpha - y) \vee 0 - (\alpha - z) \vee 0\big) \vee 0 \\ &\leq (\alpha - y - \alpha + z) \vee 0 \\ &= (z - y) \vee 0 \\ &= d_{\mathbb{P}}^{-1}(y,z), \end{aligned}$$

which proves the inequality in (i).
(ii) Take $\alpha = x + \omega$. Then we have for all $y \in [0, \infty]$

$$\begin{aligned} d_{\mathbb{P}}^{-1}(x,y) \wedge \omega &= \big((y-x) \vee 0\big) \wedge \omega \\ &\leq (y-x) \vee 0 \\ &= -(x-y) \vee 0 \\ &\leq \Big(\omega - \big((x-y)+\omega\big) \vee 0\Big) \vee 0 \\ &= d_{\mathbb{P}}^{-1}\big(T_\alpha(x), T_\alpha(y)\big). \end{aligned}$$

□

Corollary 3.4.2 *The concretely reflective hull of pq**MET**$^\infty$ is **AP**.*

Chapter 4

APPROACH PROPERTIES

In this section we will study properties of approach spaces. In view of the fact that approach spaces generalize at the same time topological spaces and metric spaces, the properties which we will study will be generalizations of similar topological or metric properties. If (P) is a topological property then we will often say that an approach space has (P) if and only if its topological coreflection has (P). This rule will not be followed for properties of metric spaces. In this section we will describe two topological properties which will follow this rule (compactness and connectedness) and one which will be treated in a completely different way (regularity).

First of all we would like to note that if a property is topological, then it is actually a property on the equivalence classes of homeomorphic spaces. The most trivial topological invariant is the topology itself, and analogously the convergence structure, closure operator and neighborhood systems are topological invariants as well. All these structures have counterparts on the approach level which quantify their topological counterparts. The distance quantifies the closure operator, the limit operator quantifies the convergence structure, and the local distances quantify the neighborhood system.

This gives us an idea how we can define invariants in approach spaces. Rather than being restricted to asserting that a space does or does not have a certain property, we will define measures of properties in a systematic way, restricting ourselves to quantifications of topological properties.

In the first two sections we will define such measures which will be quantifications of the topological properties compactness and connectedness. For these sections, we refer to the sixth chapter in the book on approach spaces by R. Lowen [12].

In the last section, we will define a notion of regularity on the approach level. This property will not follow the general rule described above, where we stated that an approach space has a topological property (P) iff its topological coreflection has (P). A counterexample will be given later on in this chapter. In contrast to the two foregoing properties, this property will not be treated by means of a measure, since there seems to be no obvious way to define one. Here we will define regularity based on a characterization of regularity in **TOP**. We will see that this definition provides us with a nice theory. We will give various characterizations

of regularity and will end this section with an interesting application of regularity by lifting an extension theorem in **TOP** to an extension theorem in **AP**.

For the definition of regularity, we refer to the doctoral dissertation of K. Robeys [14]. For the characterizations of regularity we used various works. First of all, we refer to the paper on convergence approach spaces by P. Brock and D. Kent [5]. Secondly, we used the paper about regularity in approach theory by B. Banaschewski, R. Lowen and C. Van Olmen [2]. At last, we refer to a preprint of the book on index theory by R. Lowen [13], which is scheduled to appear in 2013. For the extension theorem on the end of this chapter, we refer the reader to the paper by Jäger, titled 'Extensions of contractions and uniform contractions on dense subspaces', [8].

4.1 Compactness

In this section we will describe a notion of compactness on the approach level. To do so, we will inspire us by looking at the topological notion of compactness and the metric notion of total boundedness. Compactness is a topological property, which in the case of metrizable topological spaces is closely linked to the metric concept of total boundedness. Both properties have very similar characterizations. A characterization for compactness is given by open covers. For a topological space to be compact, given an open cover, there should exist a finite subcollection which still covers the whole space. The metric concept of total boundedness has a very similar characterization. Given a collection of balls with an arbitrary fixed radius. For a metric space to be totally bounded, there should exist a finite subcollection which still covers the whole space.

In this section we will introduce the measure of compactness for approach spaces. This will turn out to be a unifying theory behind these different but very similar concepts.

Kuratowski introduced a measure of noncompactness for subsets of a complete metric space. Since then several variants have been introduced in the literature. Probably the nicest and most widely used measure of noncompactness is the so-called ball measure or Hausdorff measure of noncompactness. This measure is defined as follows.

If (X, d) is a p-metric space and $A \subset X$, we define the **Hausdorff** or **ball measure of noncompactness** as follows

$$m_H(A) := \inf \Big\{ \epsilon \in \mathbb{R}^+ \mid \exists x_1, \cdots, x_n \in X : A \subset \bigcup_{i=1}^{n} B(x_i, \epsilon) \Big\}.$$

It differs from the original measure of noncompactness only slightly.

The value

$$m_K(A) := \inf \Big\{ \epsilon \in \mathbb{R}^+ \mid \exists X_1, \cdots, X_n \subset X : \max_{i=1}^{n} \operatorname{diam}(X_i) \leq \epsilon, A \subset \bigcup_{i=1}^{n} X_i \Big\}$$

is called the **Kuratowski measure of noncompactness**.

The following proposition gives an easy relation between the Hausdorff measure of noncompactness and the Kuratowski measure of noncompactness.

Proposition 4.1.1 *Given a p-metric space (X, d). For any $A \subset X$ we have $m_H(A) \leq m_K(A) \leq 2m_H(A)$.*

Proof. For $A \subset X$ we define the following sets:

$$H := \left\{\epsilon \in \mathbb{R}^+ \mid \exists x_1, \cdots, x_n \in X : A \subset \bigcup_{i=1}^{n} B(x_i, \epsilon)\right\}$$

and

$$K := \left\{\epsilon \in \mathbb{R}^+ \mid \exists X_1, \cdots, X_n \subset X : \max_{i=1}^{n} \operatorname{diam} X_i \leq \epsilon, A \subset \bigcup_{i=1}^{n} X_i\right\}.$$

First we prove $m_H(A) \leq m_K(A)$. To prove this part we claim that $K \subset H$. Let $\epsilon \in K$. Then there exist $X_1, \cdots, X_n \subset X$ such that $\max_{i=1}^{n} \operatorname{diam} X_i \leq \epsilon$ and $A \subset \bigcup_{i=1}^{n} X_i$. Choose for all $i \in \{1, \cdots, n\}$ an element $x_i \in X_i$. Then the following holds for every $y \in X_i$:

$$\begin{aligned} y \in X_i \quad &\Rightarrow \quad \forall x \in X_i : d(x, y) \leq \epsilon \\ &\Rightarrow \quad d(x_i, y) \leq \epsilon \\ &\Rightarrow \quad y \in B(x_i, \epsilon) \end{aligned}$$

Hence $X_i \subset B(x_i, \epsilon)$. This all means that we have $x_1, \cdots, x_n \in X$ such that

$$A \subset \bigcup_{i=1}^{n} X_i \subset \bigcup_{i=1}^{n} B(x_i, \epsilon).$$

Therefore we have $\epsilon \in H$. Since $K \subset H$ implies $\inf H \leq \inf K$, we have $m_H(A) \leq m_K(A)$.

Next we prove $m_K(A) \leq 2m_H(A)$. We claim that $2H \subset K$. Let $\epsilon \in 2H$. Then there exist $x_1, \cdots, x_n \in X$ such that $A \subset \bigcup_{i=1}^{n} B(x_i, \frac{\epsilon}{2})$. Set $X_i = B(x_i, \frac{\epsilon}{2})$, for all $i \in \{1, \cdots, n\}$. Then we have $\operatorname{diam} X_i \leq \epsilon$, for every $i \in \{1, \cdots, n\}$ and $A \subset \bigcup_{i=1}^{n} X_i$. Hence $\epsilon \in K$. Since $2H \subset K$ implies $\inf K \leq 2 \inf H$, we have $m_K(A) \leq 2m_H(A)$. □

A first remarkable observation is that both measures are metric dependent, whereas compactness is only dependent on the topology underlying the metric.

Later on in this section we show that the Hausdorff measure of noncompactness arises as a very canonical concept in the setting of approach spaces.

Definition 4.1.2 *Given an approach space (X, δ), we define the* ***measure of compactness*** *of X as*

$$\mu_c(X) := \sup_{\mathcal{F} \in \boldsymbol{F}(X)} \inf_{x \in X} \alpha \mathcal{F}(x).$$

The idea behind this definition is the following. In topological spaces compactness means that every filter has an adherence point. The value μ_c is based on the verification for all filters of what their best adherence points are.

In the following proposition we give some equivalent forms of the definition of the measure of compactness.

Proposition 4.1.3 *For any approach space (X, δ) we have*

$$\begin{aligned}
\mu_c(X) &= \sup_{\mathcal{U} \in \boldsymbol{U}(X)} \inf_{x \in X} \alpha\mathcal{U}(x) \\
&= \sup_{\mathcal{U} \in \boldsymbol{U}(X)} \inf_{x \in X} \lambda\mathcal{U}(x) \\
&= \sup_{\varphi \in \mathcal{D}^X} \inf_{Y \in 2^{(X)}} \sup_{z \in X} \inf_{x \in Y} \varphi(x)(x, z),
\end{aligned}$$

where $\mathcal{D}$ is a basis for the gauge $\mathcal{G}$ associated with δ.

Proof. We will denote the three expressions of the proposition, in order, by c_1, c_2 and c_3. It follows from proposition 1.5.17 that $c_1 = c_2$. Further it is obvious that we have $\mu_c(X) \geq c_1$. To prove $c_3 \geq \mu_c(X)$, suppose that $\mu_c(X) > r$. Then, by proposition 1.5.17, we can find an ultrafilter $\mathcal{U}$ on X such that

$$\begin{aligned}
\sup_{\psi \in \mathcal{D}^X} \inf_{x \in X} \inf_{U \in \mathcal{U}} \sup_{y \in U} \psi(x)(x, y) &= \inf_{x \in X} \sup_{d \in \mathcal{D}} \inf_{U \in \mathcal{U}} \sup_{y \in U} d(x, y) \\
&= \inf_{x \in X} \lambda\mathcal{U}(x) \\
&= \inf_{x \in X} \alpha\mathcal{U}(x) \\
&> r.
\end{aligned}$$

Consequently, there exists $\psi \in \mathcal{D}^X$ such that, for all $x \in X$,

$$\inf_{U \in \mathcal{U}} \sup_{y \in U} \psi(x)(x, y) > r.$$

Now suppose that for some $Y \in 2^{(X)}$ we have

$$\sup_{z \in X} \inf_{x \in Y} \psi(x)(x, z) \leq r.$$

Then the collection $\big\{\{\psi(x)(x, \cdot) \leq r\} \mid x \in Y\big\}$ is a finite cover of X and thus there exists some $x_0 \in Y$ such that $\{\psi(x_0)(x_0, \cdot) \leq r\} \in \mathcal{U}$. Then, however, we find that

$$\begin{aligned}
\inf_{U \in \mathcal{U}} \sup_{y \in U} \psi(x_0)(x_0, y) &\leq \sup_{y \in \{\psi(x_0)(x_0, \cdot)\}} \psi(x_0)(x_0, y) \\
&\leq r,
\end{aligned}$$

which is a contradiction. Consequently, for all $Y \in 2^{(X)}$, we have

$$\sup_{z \in X} \inf_{x \in Y} \psi(x)(x, z) > r$$

which proves that $c_3 \geq r$. From the arbitrariness of r this shows that $c_3 \geq \mu_c(X)$.

Finally, we show that $c_1 \geq c_3$. Suppose that $c_3 > r$. Then there exists $\varphi_0 \in \mathcal{D}^X$ such that, for all $Y \in 2^{(X)}$,

$$\sup_{z \in X} \inf_{x \in Y} \varphi_0(x)(x, z) > r.$$

Consequently, if, for all $Y \in 2^{(X)}$, we let

$$F_Y := \left\{ \inf_{x \in Y} \varphi_0(x)(x, \cdot) > r \right\}$$

then it follows that the collection $\{F_Y \mid Y \in 2^{(X)}\}$ is a basis for a filter $\mathcal{F}$ on X. Let $\mathcal{W}$ be an arbitrary ultrafilter finer than $\mathcal{F}$, then it follows that

$$\begin{aligned}
c_1 &= \sup_{\mathcal{U} \in \mathbf{U}(X)} \inf_{x \in X} \alpha\mathcal{U}(x) \\
&\geq \inf_{x \in X} \alpha\mathcal{W}(x) \\
&= \inf_{x \in X} \sup_{d \in \mathcal{D}} \sup_{W \in \mathcal{W}} \inf_{y \in W} d(x, y) \\
&= \sup_{\varphi \in \mathcal{D}^X} \inf_{x \in X} \sup_{W \in \mathcal{W}} \inf_{y \in W} \varphi(x)(x, y) \\
&\geq \inf_{x \in X} \sup_{W \in \mathcal{W}} \inf_{y \in W} \varphi_0(x)(x, y) \\
&\geq \inf_{x \in X} \inf_{y \in F_{\{x\}}} \varphi_0(x)(x, y) \\
&\geq r,
\end{aligned}$$

which by arbitrariness of r shows that $c_1 \geq c_3$. □

The following theorem shows the link between the topological property of compactness and the measure of compactness. We want the measure of compactness to be a generalization of the topological property compactness. When given a compact topological space, we can look at the underlying approach space and calculate the measure of compactness for this approach space. We would like this measure to be zero, since we started from a compact topological space. The following theorem shows that this is indeed true.

Theorem 4.1.4 *If $(X, \delta_{\mathcal{T}})$ is a topological approach space, then the following are equivalent:*

1. *$(X, \mathcal{T})$ is compact,*
2. *$\mu_c(X) = 0$.*

Proof. The adherence operator in topological approach spaces attains only the values 0 and ∞. By definition of μ_c and by proposition 2.1.3, we have that $\mu_c(X) = 0$ precisely means that every filter has an adherence point. □

Next we will describe the link between the metric concept of total boundedness and the measure of compactness. In the introduction we mentioned that the

measure of compactness should also be a generalization of the metric concept of total boundedness. Given a totally bounded metric space, we can look at the underlying approach space and calculate the associated measure of compactness. We would like this measure to be zero as well. The following theorem shows that this is indeed the case.

Theorem 4.1.5 *If (X, δ_d) is a p-metric approach space, then the following are equivalent:*

1. *(X, d) is totally bounded,*
2. *$\mu_c(X) = 0$.*

Proof. It follows from proposition 3.1.1 that $\mathcal{H} = \{d\}$ is a basis for the gauge of (X, δ_d). Applying proposition 4.1.3 it therefore follows that

$$\mu_c(X) = \inf_{Y \in 2^{(X)}} \sup_{z \in X} \inf_{x \in Y} d(x, z).$$

Consequently, the fact that $\mu_c(X) = 0$ means that for each $\epsilon > 0$ there exists a finite set $Y \in 2^{(X)}$ such that $X = \bigcup_{y \in Y} B(y, \epsilon)$, i.e. X is totally bounded. □

The following theorem shows that there is also an interesting relation between bounded metric spaces and the associated measure of compactness (for the underlying approach space).

Theorem 4.1.6 *If (X, δ_d) is a p-metric approach space, then the following are equivalent:*

1. *(X, d) is bounded,*
2. *$\mu_c(X) < \infty$.*

Proof. It follows again from proposition 3.1.1 that $\mathcal{H} = \{d\}$ is a basis for the gauge of (X, δ_d) and therefore we get

$$\mu_c(X) = \inf_{Y \in 2^{(X)}} \sup_{z \in X} \inf_{x \in Y} d(x, z).$$

The fact that $\mu_c(X) < \infty$ means that there exists a finite set $Y \in 2^{(X)}$ such that

$$\sup_{z \in X} \inf_{x \in Y} d(x, z) < \infty.$$

Hence, there exists $M \in \mathbb{R}^+$ such that

$$\sup_{z \in X} \inf_{x \in Y} d(x, z) \leq M.$$

This means that for every $z \in X$ there exists an element $x \in Y$ such that $d(x, z) \leq M$. Set $m = \operatorname{diam}(Y)$. Choose $y \in Y$ arbitrary. Then we get for any $z \in X$

$$\begin{aligned} z \in X \quad &\Rightarrow \quad \exists y_0 \in Y : d(y_0, z) \leq M \\ &\Rightarrow \quad d(y, z) \leq d(y, y_0) + d(y_0, z) \leq m + M \\ &\Rightarrow \quad z \in B(y, m + M), \end{aligned}$$

hence $X \subset B(y, M+m)$. This means that X is bounded. $\square$

The following theorem shows us that, where both measures are defined, the Hausdorff measure of noncompactness m_H and the measure of compactness μ_c coincide.

Theorem 4.1.7 *If (X, δ_d) is a p-metric approach space, then $\mu_c(X) = m_H(X)$.*

Proof. For any $\epsilon > 0$, we have that $\mu_c(X) < \epsilon$ if and only if there exists a finite set $Y \in 2^{(X)}$ such that for all $z \in X$ there exists $x \in Y$ such that $z \in B(x, \epsilon)$, i.e., such that $X = \bigcup_{y \in Y} B(y, \epsilon)$. Thus $\mu_c(X) < \epsilon$ if and only if $m_H(X) < \epsilon$. $\square$

The following theorem states that the measure of compactness decreases under contractions.

Theorem 4.1.8 *If (X, δ) and (X', δ') are approach spaces and the function $f : (X, \delta) \to (X', \delta')$ is a surjective contraction, then $\mu_c(X') \leq \mu_c(X)$.*

Proof. Since every ultrafilter on X' is the image by f of an ultrafilter on X, it follows from proposition 1.6.2 that

$$\begin{aligned}
\mu_c(X') &= \sup_{\mathcal{U} \in \mathbf{U}(X')} \inf_{y \in X'} \lambda' \mathcal{U}(y) \\
&\leq \sup_{\mathcal{U} \in \mathbf{U}(X)} \inf_{x \in X} \lambda' f(\mathcal{U})(f(x)) \\
&\leq \sup_{\mathcal{U} \in \mathbf{U}(X)} \inf_{x \in X} \lambda \mathcal{U}(x) \\
&\leq \mu_c(X).
\end{aligned}$$

$\square$

Thanks to this theorem we are able to acquire some well-known results concerning the image of compact topological spaces, totally bounded p-metric spaces and bounded p-metric spaces.

Corollary 4.1.9 *The continuous image of a compact topological space is compact.*

Corollary 4.1.10 *The non-expansive image of a totally bounded p-metric space is totally bounded.*

Corollary 4.1.11 *The non-expansive image of a bounded p-metric space is bounded.*

The well-known Tychonoff theorem for the product of topological spaces, will be a result of the following theorem.

Theorem 4.1.12 *If $(X_j, \delta_j)_{j \in J}$ is a family of approach spaces, then*

$$\mu_c\left(\prod_{j \in J} X_j\right) = \sup_{j \in J} \mu_c(X_j).$$

Proof. It follows from theorem 1.7.1 that if $\mathcal{U}$ is an ultrafilter on $\prod_{j\in J} X_j$, then $\alpha\mathcal{U} = \sup_{j\in J} \alpha_j(\mathrm{pr}_j(\mathcal{U})) \circ \mathrm{pr}_j$. The proof of this theorem now follows from the following calculations:

$$\begin{aligned}
\mu_c\Big(\prod_{j\in J} X_j\Big) &= \sup_{\mathcal{U}\in\mathbf{U}(\prod_{j\in J} X_j)} \inf_{x\in\prod_{j\in J} X_j} \alpha\mathcal{U}(x) \\
&= \sup_{\mathcal{U}\in\mathbf{U}(\prod_{j\in J} X_j)} \inf_{x\in\prod_{j\in J} X_j} \sup_{j\in J} \alpha_j(\mathrm{pr}_j(\mathcal{U}))(x_j) \\
&= \sup_{\mathcal{U}\in\mathbf{U}(\prod_{j\in J} X_j)} \sup_{j\in J} \inf_{z\in X_j} \alpha_j(\mathrm{pr}_j(\mathcal{U}))(z) \\
&= \sup_{j\in J} \sup_{\mathcal{U}\in\mathbf{U}(X_j)} \inf_{z\in X_j} \alpha_j\mathcal{U}(z) \\
&= \sup_{j\in J} \mu_c(X_j).
\end{aligned}$$

□

Corollary 4.1.13 *(**Tychonoff theorem**) The product of a family of topological spaces is compact if and only if each factor space is compact.*

Corollary 4.1.14 *The product of a finite family of p-metric spaces is totally bounded if and only if each factor space is totally bounded.*

We end this section about compactness with some examples. We will investigate the measures of compactness for the approach spaces introduced in example 1.5.14.

Example 4.1.15

1. $\mu_c(\mathbb{E}) = \infty$
   Proof. $\mathcal{D} = \{d_{\mathbb{E}}\}$ is a basis for the gauge of $\mathbb{E}$. We have that $d_{\mathbb{E}}(x, \infty) = \infty$, for every $x \in [0, \infty[$. By applying theorem 4.1.6, we get $\mu_c(\mathbb{E}) = \infty$. □

2. $\mu_c(\mathbb{P}) = 0$
   Proof. By proposition 4.1.3 we have

   $$\mu_c(\mathbb{P}) = \sup_{\mathcal{U}\in\mathbf{U}([0,\infty])} \inf_{x\in[0,\infty]} \lambda\mathcal{U}(x).$$

   In example 1.5.14 we proved that $\lambda\mathcal{U}(x) = \big(x - l(\mathcal{U})\big) \vee 0$, where $l(\mathcal{U}) = \inf_{U\in\mathrm{sec}(\mathcal{U})} \sup U$. We know that $l(\mathcal{U}) \geq 0$, since every $U \in \mathcal{U}$ is a subset of $[0, \infty]$. For every $\mathcal{U} \in \mathbf{U}([0, \infty])$, we have

   $$\inf_{x\in[0,\infty]} \lambda\mathcal{U}(x) = \inf_{x\in[0,\infty]} (x - l(\mathcal{U})) \vee 0.$$

   For $x = 0$, we have $(x - l(\mathcal{U})) \vee 0 = 0$. This implies that $\inf_{x\in[0,\infty]} \lambda\mathcal{U}(x) = 0$ and hence $\mu_c(\mathbb{P}) = 0$. □

4.2 Connectedness

In this section we will describe a notion of connectedness on the approach level. Just like in the previous section, we will look at a topological property and a metric concept which have similar characterizations. Connectedness is a well known topological property. Topological connectedness of a space X can be characterized by the fact that X cannot be split into two nontrivial closed parts. The metric notion of Cantor-connectedness, introduced by Cantor himself, has a very similar characterization. A Cantor-connected metric space is characterized by the fact that the space cannot be split into two nontrivial parts which lie at a strictly positive distance from each other.

In this section we will introduce the measure of connectedness for approach spaces. This will turn out to be a unifying theory behind these different but very similar concepts.

In order to formulate our definition of a measure of connectedness we need some preliminary concepts.

For each $\epsilon > 0$, we define the space D_ϵ to be the two point set $\{0, \infty\}$ equipped with the ∞p-metric d_ϵ, where $d_\epsilon(0, \infty) := \epsilon$. The space D_ϵ serves as a prototype for a disconnected space and the disconnectedness is quantified by the distance between the points 0 and ∞.

Definition 4.2.1 *An approach space (X, δ) is said to be **ϵ-connected** if the only contractions from (X, δ) to D_ϵ are constant functions.*

*Given an approach space (X, δ), we define the **measure of connectedness** of X as*

$$\mu_{cn}(X) := \inf\{\epsilon > 0 \mid X \text{ is } \epsilon\text{-connected }\}.$$

Proposition 4.2.2 *If (X, δ) is an approach space which is ϵ-connected and $\epsilon' > \epsilon$, then (X, δ) is ϵ'-connected.*

Proof. Let $f : (X, \delta) \to D_{\epsilon'}$ be a contraction. We have for any $x \in X$ and $A \subset X$

$$\begin{aligned} \delta_{d_\epsilon}(f(x), f(A)) &\leq \delta_{d_{\epsilon'}}(f(x), f(A)) \\ &\leq \delta(x, A). \end{aligned}$$

Hence $f : (X, \delta) \to D_\epsilon$ is a contraction. Therefore f is constant. □

Example 4.2.3

The space D_ϵ is ϵ'-connected, for all $\epsilon' > \epsilon$, but is not ϵ-connected. Consequently $\mu_{cn}(D_\epsilon) = \epsilon$.

Proof. Suppose $\epsilon' > \epsilon$ and $f : D_\epsilon \to D_{\epsilon'}$ is a contraction. We have to show that $f(0) = f(\infty)$. Suppose $f(0) \neq f(\infty)$. Then we have

$$\epsilon' = d_{\epsilon'}(f(0), f(\infty)) \leq d_\epsilon(0, \infty) = \epsilon,$$

which is a contradiction. This proves that f is constant.

D_ϵ is not ϵ-connected, since

$$\begin{array}{rccc} f: & D_\epsilon & \to & D_\epsilon \\ & 0 & \mapsto & \infty \\ & \infty & \mapsto & 0 \end{array}$$

is a contraction, which is not constant. □

In the following theorem we describe the link with topological spaces. Since topological spaces can be seen as approach spaces, we would like the measure of connectedness to be zero whenever the approach space is derived from a connected topological space. The next theorem shows that this is indeed true.

Theorem 4.2.4 *If $(X, \delta_\mathcal{T})$ is a topological approach space, then the following are equivalent:*

1. *$(X, \mathcal{T})$ is connected,*
2. *$\mu_{cn}(X) = 0$.*

Proof. To see that the first property implies the second one, suppose that for some $\epsilon > 0$ X is not ϵ-connected. Then there exists a contraction $f : X \to D_\epsilon$ which is not constant. Let $X_0 := f^{-1}(\{0\})$ and $X_\infty := f^{-1}(\{\infty\})$. For any $x \in X_0$, it follows that $\epsilon \leq \delta_\mathcal{T}(x, X_\infty)$, which implies that $x \notin \mathrm{cl}_\mathcal{T}(X_\infty)$. Thus $X_0 \cap \mathrm{cl}_\mathcal{T}(X_\infty) = \emptyset$. Analogously $X_\infty \cap \mathrm{cl}_\mathcal{T}(X_0) = \emptyset$. This proves that both X_0 and X_∞ are open and thus $(X, \mathcal{T})$ is not connected.

To see that the second property implies the first one, let X be partitioned into two open sets X_0 and X_∞. Then we have that, for any $\epsilon > 0$, the map $f : X \to D_\epsilon$ defined by $f(X_0) := \{0\}$ and $f(X_\infty) := \{\infty\}$ is a contraction. Consider $x \in X$ and $A \subset X$. There are three possibilities; $f(A)$ can be equal to D_ϵ, $\{0\}$ or $\{\infty\}$. First suppose that $f(A) = D_\epsilon$. Then we get $\delta_{d_\epsilon}(f(x), f(A)) = 0 \leq \delta_\mathcal{T}(x, A)$, which shows that f is indeed a contraction. Secondly, suppose that $f(A) = \{0\}$. If $x \in X_0$, then we have $\delta_{d_\epsilon}(f(x), f(A)) = 0 \leq \delta_\mathcal{T}(x, A)$. Otherwise, if $x \in X_\infty$, then we have $\delta_{d_\epsilon}(f(x), f(A)) = d_\epsilon(\infty, 0) = \epsilon \leq \delta_\mathcal{T}(x, A)$. We further explain the last step. If $\delta_\mathcal{T}(x, A) < \epsilon$, then $\delta_\mathcal{T}(x, A) = 0$. This implies that $x \in \mathrm{cl}_\mathcal{T}(A)$, hence $f(x) = 0$. This, however, is a contradiction. This shows that in this case f is a contraction as well. Consider now the third possibility, $f(A) = \{\infty\}$. This part of the proof, however, is completely similar to the previous step. Thus $\mu_{cn}(X) = \infty$. □

Before we give a result about the measure of connectedness in p-metric approach spaces, we must define the notion of Cantor-connectedness in p-metric spaces.

Definition 4.2.5 *A p-metric space (X, d) is **Cantor-connected** if for any $\epsilon > 0$, any two points $x, y \in X$ can be connected by an ϵ-chain. This means that there exist $x_0 = x, x_1, \cdots, x_n = y$ such that for all $i \in \{1, \cdots, n\} : d(x_{i-1}, x_i) \leq \epsilon$.*

The following characterization of Cantor-connectedness will be easier to work with in the following theorem.

Proposition 4.2.6 *A space (X, d) is Cantor-connected if and only if it cannot be partitioned into two sets A and B such that $d(A, B) > 0$.*

Proof. We first prove that if (X, d) is Cantor-connected, X cannot be partitioned into two sets A and B such that $d(A, B) > 0$. Suppose that there exists a partition of X into two sets A and B with $d(A, B) > 0$. Take $a \in A$ and $b \in B$, take $\epsilon > 0$. By the fact that X is Cantor-connected, there exist $a = x_0, x_1, \cdots, x_n = b$ in X such that for all $i \in \{1, \cdots, n\}$ $d(x_{i-1}, x_i) \leq \epsilon$. But then we get

$$d(a, b) \leq \sum_{i=1}^{n} d(x_{i-1}, x_i) \leq n\epsilon.$$

By arbitrariness of ϵ we get that $d(a, b) = 0$, hence $d(A, B) = 0$. This is a contradiction.

Now we show that if X cannot be partitioned into two sets A and B with $d(A, B) > 0$, X is Cantor-connected. Take $x \in X$ and consider the set

$$A = \{y \in X \mid \ y \text{ can be reached from } x \text{ by finitely many steps each of length smaller than } \epsilon\}.$$

We have to show that $A = X$. Suppose that $A \neq X$. Then A and $X \setminus A$ form a partition of X and $d(A, X \setminus A) > 0$, which is a contradiction. □

The next theorem shows the relation between the measure of connectedness on approach spaces and the notion of Cantor-connectedness in p-metric spaces. Since p-metric spaces can be seen as approach spaces, we would like the measure of connectedness to be zero whenever the approach space is derived from a Cantor-connected p-metric space. The next theorem shows that this is indeed true.

Theorem 4.2.7 *If (X, δ) is a p-metric space, then the following are equivalent:*

1. *(X, d) is Cantor-connected,*
2. *$\mu_{cn}(X) = 0$.*

Proof. To see that the first property implies the second one, suppose that for some $\epsilon > 0$ X is not ϵ-connected. Then there exists a contraction $f : X \to D_\epsilon$ which is not constant. Let $X_0 := f^{-1}(\{0\})$ and $X_\infty := f^{-1}(\{\infty\})$. Then it follows that

$$\begin{aligned} d(X_0, X_\infty) &= \inf_{x \in X_0} \inf_{y \in X_\infty} d(x, y) \\ &\geq \inf_{x \in X_0} \inf_{y \in X_\infty} d_\epsilon(f(x), f(y)) \\ &= \inf_{x \in X_0} \inf_{y \in X_\infty} d_\epsilon(0, \infty) \\ &= \epsilon \\ &> 0. \end{aligned}$$

To see that the second property implies the first one, let X be partitioned into two sets X_0 and X_∞ such that $d(X_0, X_\infty) \geq \epsilon$. Then the map $f : X \to D_\epsilon$ defined by $f(X_0) := \{0\}$ and $f(X_\infty) := \{\infty\}$ is a contraction and thus $\mu_{cn}(X) \geq \epsilon > 0$. □

Examples 4.2.8

We give two examples to illustrate the measure of connectedness in a topological and a metric setting.

1. Consider $\mathbb{Q}$ to be equipped with the Euclidean topology. This space is not connected, hence $\mu_{cn}(\mathbb{Q}) = \infty$. If we consider $\mathbb{Q}$ to be equipped with the Euclidean metric, then it is Cantor-connected and $\mu_{cn}(\mathbb{Q}) = 0$.

2. Consider $\mathbb{N}$ with the Euclidean topology. Then clearly $\mathbb{N}$ is not connected, hence $\mu_{cn}(\mathbb{N}) = \infty$. If we consider $\mathbb{N}$ to be equipped with the Euclidean metric, then it is also not Cantor-connected. In this case the measure of connectedness is $\mu_{cn}(\mathbb{N}) = 1$.
   Proof. First we prove that for all $\epsilon > 1$, $\mathbb{N}$ is ϵ-connected. Consider a contraction $f : (\mathbb{N}, d_E) \to D_\epsilon$. Set $X_0 := f^{-1}(\{0\}), X_\infty := f^{-1}(\{\infty\})$. We have to prove that one of these sets is empty. Suppose that $X_0 \neq \emptyset$ and $X_\infty \neq \emptyset$. Then there exists $n \in X_0$ such that $n+1 \notin X_0$ or $n-1 \notin X_0$. This follows from the fact that if for every $n \in X_0$ we have $n+1 \in X_0$ and $n-1 \in X_0$, then we would have $X_0 = \mathbb{N}$ and $X_\infty = \emptyset$, which is a contradiction. Hence suppose that $n+1 \notin X_0$, then we have

   $$\begin{aligned} 1 < \epsilon &= d_\epsilon(f(n), f(n+1)) \\ &\leq d_E(n, n+1) \\ &= 1, \end{aligned}$$

   which is in contradiction with the choice of ϵ. In the other case, where $n-1 \notin X_0$, we come to the same conclusion. Hence $X_0 = \emptyset$ or $X_\infty = \emptyset$. This proves that f is constant.

   We now prove that for $\epsilon < 1$ $\mathbb{N}$ is not ϵ-connected. Consider the function $f : \mathbb{N} \to D_\epsilon$ where

   $$f(n) := \begin{cases} 0 & n \text{ even}, \\ \infty & n \text{ odd}. \end{cases}$$

   For any $n, m \in \mathbb{N}$, we have

   - If $n, m \in 2\mathbb{N}$:
     $d_\epsilon(f(n), f(m)) = d_\epsilon(0, 0) = 0 \leq d_E(n, m)$.

   - If $n, m \in 2\mathbb{N} + 1$:
     $d_\epsilon(f(n), f(m)) = d_\epsilon(\infty, \infty) = 0 \leq d_E(n, m)$.

   - If $n \in 2\mathbb{N}$, $m \in 2\mathbb{N} + 1$:
     $d_\epsilon(f(n), f(m)) = d_\epsilon(0, \infty) = \epsilon < 1 \leq d_E(n, m)$.

   Then, for any $n \in \mathbb{N}$ and $A \subset \mathbb{N}$, we have

   $$\delta_{d_\epsilon}(f(n), f(A)) = \inf_{a \in A} d_\epsilon(f(n), f(a)) \leq \inf_{a \in A} d_E(n, a) = \delta_{d_E}(n, A).$$

   Hence $f : (\mathbb{N}, \delta_{d_E}) \to D_\epsilon$ is a contraction which is not constant. $\square$

In the following proposition, we give an alternative characterization of the measure of connectedness.

Proposition 4.2.9 *For any approach space (X, δ), we have*

$$\mu_{cn}(X) = \sup_{A \in 2^X, A \neq \emptyset, A \neq X} \min \left\{ \inf_{x \in A} \delta(x, X \setminus A), \inf_{x \in X \setminus A} \delta(x, A) \right\}.$$

Proof. That X is not ϵ-connected is equivalent to the existence of a non-constant contraction $f : X \to D_\epsilon$. This in turn is equivalent with the existence of a set A such that $A \neq \emptyset$, $A \neq X$, and such that, for all $x \in X$, $\delta(x, X \setminus A) \geq \epsilon$ and, for all $x \in X \setminus A$, $\delta(x, A) \geq \epsilon$. □

The following proposition shows that ϵ-connectedness is preserved under surjective contractions.

Proposition 4.2.10 *If (X, δ) and (X', δ') are approach spaces, (X, δ) is ϵ-connected, and $f : (X, \delta) \to (X', \delta')$ is a surjective contraction, then (X', δ') is ϵ-connected.*

Proof. Suppose that X is ϵ-connected and that $h : X' \to D_\epsilon$ is a contraction. Then $h \circ f : X \to D_\epsilon$ is also a contraction, which is therefore constant. Then it follows from the surjectivity of f that h too is constant. This proves that X' is ϵ-connected. □

The next theorem states that the measure of connectedness decreases under surjective contractions.

Theorem 4.2.11 *If (X, δ) and (X', δ') are approach spaces and the function $f : (X, \delta) \to (X', \delta')$ is a surjective contraction, then $\mu_{cn}(X') \leq \mu_{cn}(X)$.*

Proof. This follows from the definition of μ_{cn} and the previous proposition. □

Thanks to this theorem we are able to acquire some well-known results concerning the image of connected topological spaces and Cantor-connected p-metric spaces.

Corollary 4.2.12 *The continuous image of a connected topological space is connected.*

Corollary 4.2.13 *The non-expansive image of a Cantor-connected p-metric space is Cantor-connected.*

In order to give a product theorem for the measure of connectedness we need some preliminary results, which are interesting in their own right.

Proposition 4.2.14 *If $(X, \delta) \in |\boldsymbol{AP}|, Y \subset Z \subset \overline{Y} \subset X$, and Y is ϵ-connected, then Z is ϵ-connected.*

Proof. Let $f : Z \to D_\epsilon$ be a contraction. Then $f_{|Y}$ is constant and thus there exists $p \in D_\epsilon$ such that $f(Y) = \{p\}$. Now, if $z \in Z$, then $\delta(z, Y) = 0$ and, hence, $\delta_{d_\epsilon}(f(z), \{p\}) \leq \delta(z, Y) = 0$, i.e. $\delta_{d_\epsilon}(f(z), \{p\}) = 0$ and thus $f(z) = p$. This gives us that f is constant. □

Definition 4.2.15 *If $\mathcal{A}$ is a collection of subsets of a set X, then $\mathcal{A}$ is said to be* ***chained*** *if, for any pair of sets $A, B \in \mathcal{A}$, there exists a finite collection $\{A_1, \cdots, A_n\} \subset \mathcal{A}$ such that $A_1 = A$, $A_n = B$ and for all $i \in \{1, \cdots, n-1\}$ $A_i \cap A_{i+1} \neq \emptyset$.*

Proposition 4.2.16 *Let (X, δ) be an approach space and let $\mathcal{A}$ be a chained collection of subsets of X. If each $A \in \mathcal{A}$ is ϵ_A-connected, then $\bigcup \mathcal{A}$ is $\big(\sup_{A \in \mathcal{A}} \epsilon_A\big)$-connected.*

Proof. Suppose that $Y := \bigcup \mathcal{A}$ is not $\big(\sup_{A \in \mathcal{A}} \epsilon_A\big)$-connected. Then there exists a subset $B \subset Y$ such that $\emptyset \neq B \neq Y$ and such that, if we let $\epsilon := \sup_{A \in \mathcal{A}} \epsilon_A$,

$$\inf_{x \in B} \delta(x, Y \setminus B) \geq \epsilon \text{ and } \inf_{x \in Y \setminus B} \delta(x, B) \geq \epsilon.$$

Now suppose that there exists a set $A \in \mathcal{A}$ such that $A \nsubseteq B$ and $A \nsubseteq Y \setminus B$. Then it follows that

$$\inf_{x \in A \cap B} \delta(x, A \setminus B) \geq \inf_{x \in B} \delta(x, Y \setminus B) \geq \epsilon \geq \epsilon_A$$

and

$$\inf_{x \in A \setminus B} \delta(x, A \cap B) \geq \inf_{x \in Y \setminus B} \delta(x, B) \geq \epsilon \geq \epsilon_A,$$

which is impossible. Consequently, if we let $\mathcal{C} := \{A \in \mathcal{A} \mid A \subset B\}$ and $\mathcal{D} := \{A \in \mathcal{A} \mid A \subset Y \setminus B\}$, then $\mathcal{A} = \mathcal{C} \cup \mathcal{D}$. Since $\mathcal{A}$ is chained, this implies that either $\mathcal{C} = \emptyset$ or $\mathcal{D} = \emptyset$. Suppose, for example, that $\mathcal{D} = \emptyset$, then $\mathcal{A} = \mathcal{C}$ and it follows that $Y = B$ which is a contradiction. □

Proposition 4.2.17 *If $(X_j, \delta_j)_{j \in J}$ is a family of approach spaces such that for all $j \in J$, (X_j, δ_j) is ϵ_j-connected, then $\prod_{j \in J}(X_j, \delta_j)$ is $\big(\sup_{j \in J} \epsilon_j\big)$-connected.*

Proof. Let $(z_j)_{j \in J}$ be a fixed point in the product space $\prod_{j \in J} X_j$. For any subset $K \subset J$, let

$$X_K := \big\{(x_j)_{j \in J} \mid x_i = z_i \quad \forall i \in J \setminus K\big\}.$$

By induction on the cardinality of K, making use of proposition 4.2.16, one sees that, for any finite set K, X_K is $\big(\sup_{k \in K} \epsilon_k\big)$-connected. Now the collection

$$\mathcal{B} := \{X_K \mid K \subset J, K \text{ finite}\}$$

is chained, and hence, it once again follows from proposition 4.2.16 that $\bigcup \mathcal{B}$ is $\big(\sup_{j \in J} \epsilon_j\big)$-connected. Since $\bigcup \mathcal{B}$ is dense in $\prod_{j \in J} X_j$ it follows from proposition 4.2.14 that $\prod_{j \in J} X_j$ is $\big(\sup_{j \in J} \epsilon_j\big)$-connected. □

Theorem 4.2.18 *If $(X_j, \delta_j)_{j\in J}$ is a family of approach spaces, then*

$$\mu_{cn}\left(\prod_{j\in J} X_j\right) = \sup_{j\in J} \mu_{cn}(X_j).$$

Proof. That $\mu_{cn}(\prod_{j\in J} X_j) \geq \sup_{j\in J} \mu_{cn}(X_j)$ follows from proposition 4.2.11. The converse inequality follows from proposition 4.2.17 since we have

$$\begin{aligned}
\mu_{cn}(\prod_{j\in J} X_j) &= \inf\left\{\epsilon \mid \prod_{j\in J} X_j \text{ is } \epsilon\text{-connected }\right\} \\
&\leq \inf \bigcap_{j\in J} \{\epsilon \mid X_j \text{ is } \epsilon\text{-connected }\} \\
&= \sup_{j\in J} \mu_{cn}(X_j).
\end{aligned}$$

□

Corollary 4.2.19 *The product of a family of topological spaces is connected if and only if each factor space is connected.*

Corollary 4.2.20 *The product of a finite family of p-metric spaces is Cantor-connected if and only if each factor space is Cantor-connected.*

Example 4.2.21

Consider the Cantorset C. If C is equipped with the Euclidean topology, then it is not connected. If C is equipped with the Euclidean metric, then it is not Cantor-connected. We get that $\mu_{cn}(C) = \frac{1}{3}$.
If we identify C with the product $\{0,1\}^{\mathbb{N}}$ and equip $\{0,1\}$ with the Euclidean metric, then we can also equip C with the product distance. We then get

$$\mu_{cn}(C) = \sup_{n\in\mathbb{N}} \mu_{cn}(\{0,1\}) = 1.$$

We end this section about connectedness with some examples. We will investigate the measures of connectedness for the approach spaces introduced in example 1.5.14.

Example 4.2.22

1. $\mu_{cn}(\mathbb{E}) = \infty$
   Proof. We will use the characterization of the measure of connectedness introduced in proposition 4.2.9.

$$\begin{aligned}
\mu_{cn}(\mathbb{E}) &= \sup_{\emptyset\neq A\subsetneq[0,\infty]} \min\left\{\inf_{x\in A} \delta_{\mathbb{E}}(x, [0,\infty]\setminus A), \inf_{x\in[0,\infty]\setminus A} \delta_{\mathbb{E}}(x, A)\right\} \\
&\geq \min\left\{\inf_{x\in\{\infty\}} \delta_{\mathbb{E}}(x, [0,\infty[), \inf_{x\in[0,\infty[} \delta_{\mathbb{E}}(x, \{\infty\})\right\}
\end{aligned}$$

First of all we have

$$\inf_{x\in\{\infty\}} \delta_{\mathbb{E}}(x,[0,\infty[) = \delta_{\mathbb{E}}(\infty,[0,\infty[) = \infty.$$

Secondly, we have

$$\inf_{x\in[0,\infty[} \delta_{\mathbb{E}}(x,\{\infty\}) = \inf_{x\in[0,\infty[} \inf_{a\in\{\infty\}\cap\mathbb{R}^+} |x-a| = \inf_{x\in[0,\infty[} \inf_{a\in\emptyset} |x-a| = \infty.$$

So we get

$$\mu_{cn}(\mathbb{E}) = \min\{\infty,\infty\} = \infty.$$

□

2. $\mu_{cn}(\mathbb{P}) = 0$
Proof. By proposition 4.2.9. we have

$$\mu_{cn}(\mathbb{P}) = \sup_{\emptyset\neq A\subsetneq[0,\infty]} \min\Big\{ \inf_{x\in A} \delta_{\mathbb{P}}(x,[0,\infty]\setminus A), \inf_{x\in[0,\infty]\setminus A} \delta_{\mathbb{P}}(x,A)\Big\}.$$

We notice that $A \neq \emptyset$ implies $[0,\infty]\setminus A \neq [0,\infty]$ and $A \neq [0,\infty]$ implies $[0,\infty]\setminus A \neq \emptyset$.
We have

$$\inf_{x\in A} \delta_{\mathbb{P}}(x,[0,\infty]\setminus A) = \inf_{x\in A} \big(x-\sup([0,\infty]\setminus A)\big)\vee 0 = \big(\inf A-\sup([0,\infty]\setminus A)\big)\vee 0$$

and

$$\inf_{x\in[0,\infty]\setminus A} \delta_{\mathbb{P}}(x,A) = \inf_{x\in[0,\infty]\setminus A} \big(x-\sup A\big)\vee 0 = \big(\inf([0,\infty]\setminus A)-\sup A\big)\vee 0.$$

For every $\emptyset \neq A \subsetneq [0,\infty]$, we have $\infty \in A$ or $\infty \in [0,\infty]\setminus A$. Suppose that $\infty \in A$. Then we have $\big(\inf([0,\infty]\setminus A)-\sup A\big)\vee 0 = 0$, and hence $\min\{\inf_{x\in A}\delta_{\mathbb{P}}(x,[0,\infty]\setminus A), \inf_{x\in[0,\infty]\setminus A}\delta_{\mathbb{P}}(x,A)\} = 0$. If $\infty \in [0,\infty]\setminus A$, we get $\big(\inf A-\sup([0,\infty]\setminus A)\big)\vee 0 = 0$, and hence $\min\{\inf_{x\in A}\delta_{\mathbb{P}}(x,[0,\infty]\setminus A), \inf_{x\in[0,\infty]\setminus A}\delta_{\mathbb{P}}(x,A)\} = 0$. This all implies $\mu_{cn}(\mathbb{P}) = 0$. □

4.3 Regularity

In this section we will generalize the topological concept of regularity to the realm of approach theory. As mentioned in the introduction of this chapter, we will not follow the general rule where an approach space is considered to have a topological property (P) iff the topological coreflection of the approach space has (P). We will use the definition of regularity for approach spaces introduced in the doctoral dissertation of K. Robeys [14]. This definition is inspired by a characterization of regularity in **TOP**. Later on in this section, we will construct an approach space which is not regular, but which has a regular topological coreflection. This gives us a counterexample which shows that in this case the general rule does not hold. After introducing the concept of regularity, we will give various characterizations using the limit operator. For these characterizations, we reached for the paper on

convergence approach spaces by P. Brock and D. Kent [5]. We will also give some characterization theorems using the gauge. These theorems can be found in the book on index theory by R. Lowen [13] and the paper on regularity in approach theory by B. Banaschewski, R. Lowen and C. Van Olmen [2].

We end this section with an interesting application. We will generalize a well-known extension theorem, which we can find in Bourbaki [4], so that it becomes applicable on approach spaces. To this end we refer to the paper by Jäger on extension theorems [8].

Before focusing on regularity in approach spaces, we will give a result for topological spaces. This result will inspire us for a definition of regularity in approach spaces.

For a filter $\mathcal{F}$ we define $\overline{\mathcal{F}}$ as the filter stack$\{\overline{F} \mid F \in \mathcal{F}\}$.

Proposition 4.3.1 *In a topological space $(X, \mathcal{T})$, the following properties are equivalent:*

1. *X is regular,*
2. *$\forall \mathcal{F} \in \boldsymbol{F}(X)$: $\mathcal{F} \to x \Rightarrow \overline{\mathcal{F}} \to x$,*
3. *$\forall \mathcal{U} \in \boldsymbol{U}(X)$: $\mathcal{U} \to x \Rightarrow \overline{\mathcal{U}} \to x$.*

Proof. 1) $\Rightarrow$ 2): If X is regular, then there exists a basis $\mathcal{B}$ of $\mathcal{V}(x)$ such that every set $B \in \mathcal{B}$ is closed. Consider $\mathcal{F} \in F(X)$ such that $\mathcal{F} \to x$. Then we have

$$\begin{aligned}
&\forall V \in \mathcal{V}(x),\ \exists F \in \mathcal{F} : F \subset V \\
&\Rightarrow \forall B \in \mathcal{B},\ \exists F \in \mathcal{F} : F \subset B \\
&\Rightarrow \forall B \in \mathcal{B},\ \exists F \in \mathcal{F} : \overline{F} \subset \overline{B} = B \\
&\Rightarrow \forall V \in \mathcal{V}(x),\ \exists F \in \mathcal{F} : \overline{F} \subset V.
\end{aligned}$$

This means that $\overline{\mathcal{F}} \to x$, which proves this implication.

2) $\Rightarrow$ 1): In every topological space, we have that $\mathcal{V}(x) \to x$. By applying 2), we get $\overline{\mathcal{V}(x)} \to x$. This means that $\forall V \in \mathcal{V}(x) \exists W \in \mathcal{V}(x) : \overline{W} \subset V$. Therefore, X is regular.

2) $\Rightarrow$ 3): This is trivial.

3) $\Rightarrow$ 2): Take $\mathcal{F} \in \mathbf{F}(X)$ such that $\mathcal{F} \to x$. Suppose that $\overline{\mathcal{F}}$ does not converge to x. Then there exists $V \in \mathcal{V}(x)$ such that for all $F \in \mathcal{F}$ we have that $\overline{F} \nsubseteq V$. If $\mathcal{U} \in \mathbf{U}(\mathcal{F})$, then we have $\mathcal{U} \to x$ and by 3) $\overline{\mathcal{U}} \to x$. For every $\mathcal{U} \in \mathbf{U}(\mathcal{F})$, there exists $\sigma(\mathcal{U}) \in \mathcal{U}$ such that $\overline{\sigma(\mathcal{U})} \subset V$. Apply proposition 1.2.2. Then there exists a finite set $U_\sigma \subset \mathbf{U}(\mathcal{F})$ such that

$$\bigcup_{\mathcal{U} \in U_\sigma} \sigma(\mathcal{U}) \in \mathcal{F}.$$

But then we have

$$\overline{\bigcup_{\mathcal{U} \in U_\sigma} \sigma(\mathcal{U})} \nsubseteq V.$$

Since

$$\overline{\bigcup_{\mathcal{U}\in U_\sigma} \sigma(\mathcal{U})} = \bigcup_{\mathcal{U}\in U_\sigma} \overline{\sigma(U)},$$

we get

$$\bigcup_{\mathcal{U}\in U_\sigma} \overline{\sigma(U)} \nsubseteq V,$$

which is a contradiction. □

For a filter $\mathcal{F}$, we define $\mathcal{F}^{(\gamma)}$, for any $\gamma \in [0,\infty]$, as the filter stack$\{F^{(\gamma)} \mid F \in \mathcal{F}\}$.

The following theorem will allow us to give a definition of regularity in **AP**.

Proposition 4.3.2 *For an approach space (X,λ) the following are equivalent:*

1. *For $\mathcal{F} \in \boldsymbol{F}(X)$, for $\gamma \in [0,\infty]$, and for $x \in X$ we have*

$$\lambda\mathcal{F}^{(\gamma)}(x) \leq \lambda\mathcal{F}(x) + \gamma,$$

2. *For $\mathcal{W},\mathcal{U} \in \boldsymbol{U}(X)$ and for $\gamma \in [0,\infty]$ we have*

$$\mathcal{W}^{(\gamma)} \subseteq \mathcal{U} \Rightarrow \lambda\mathcal{U}(x) \leq \lambda\mathcal{W}(x) + \gamma.$$

Proof. 1) ⇒ 2): Consider $\mathcal{W},\mathcal{U} \in \mathbf{U}(X)$ and $\gamma \in [0,\infty]$ such that $\mathcal{W}^{(\gamma)} \subseteq \mathcal{U}$. Then we have

$$\lambda\mathcal{U}(x) \leq \lambda\mathcal{W}^{(\gamma)}(x) \leq \lambda\mathcal{W}(x) + \gamma.$$

2) ⇒ 1): Let $\mathcal{F}$ be a filter on X, $\gamma \in [0,\infty]$ and $\mathcal{U} \in \mathbf{U}(X)$ such that $\mathcal{F}^{(\gamma)} \subseteq \mathcal{U}$. First we prove that there exists an ultrafilter $\mathcal{W} \in \mathbf{U}(X)$ such that $\mathcal{F} \subseteq \mathcal{W}$ and $\mathcal{W}^{(\gamma)} \subseteq \mathcal{U}$. If this is not the case, let $\{\mathcal{W}_i \mid i \in I\}$ be the collection of all ultrafilters on X refining $\mathcal{F}$. In every $\mathcal{W}_i$ we choose $W_i \in \mathcal{W}_i$ such that $W_i^{(\gamma)} \notin \mathcal{U}$. Then there exists a finite subset $\{W_{i_1},\cdots,W_{i_n}\}$ with $W_{i_j} \in \mathcal{W}_{i_j}$ and $W_{i_1} \cup \cdots \cup W_{i_n} \in \mathcal{F}$. It follows that $(W_{i_1} \cup \cdots \cup W_{i_n})^{(\gamma)} \in \mathcal{U}$. Now we have

$$\begin{aligned} x \in (W_{i_1} \cup \cdots \cup W_{i_n})^{(\gamma)} &\Leftrightarrow \delta(x, W_{i_1} \cup \cdots \cup W_{i_n}) \leq \gamma \\ &\Leftrightarrow \min_{j=1}^{n} \delta(x, W_{i_j}) \leq \gamma \\ &\Rightarrow \exists j \in \{1,\cdots,n\} : \delta(x, W_{i_j}) \leq \gamma \\ &\Rightarrow \exists j \in \{1,\cdots,n\} : x \in W_{i_j}^{(\gamma)}. \end{aligned}$$

Hence one of the $W_{i_j}^{(\gamma)} \in \mathcal{U}$, which is a contradiction.
For $\mathcal{F} \subseteq \mathcal{W}$ and $\mathcal{W}^{(\gamma)} \subseteq \mathcal{U}$ we now have

$$\begin{aligned} \lambda\mathcal{F}^{(\gamma)}(x) &\leq \lambda\mathcal{U}(x) \\ &\leq \lambda\mathcal{W}(x) + \gamma \\ &\leq \lambda\mathcal{F}(x) + \gamma, \end{aligned}$$

and since this holds for every $\mathcal{U} \in \mathbf{U}(X)$ with $\mathcal{F}^{(\gamma)} \subseteq \mathcal{U}$, we are done. □

Definition 4.3.3 *An approach space is called* ***regular*** *if it satisfies one and hence both conditions in proposition 4.3.2.*

In the following proposition we describe the relations between regularity in approach spaces and regularity in topological spaces.

Proposition 4.3.4 *1. If (X, λ) is a regular approach space, then the topological coreflection $(X, \mathcal{T}_\lambda)$ is regular as well.*

2. If $(X, \mathcal{T})$ is a regular topological space, then the underlying approach space $(X, \lambda_\mathcal{T})$ is regular as an approach space.

Proof.

1. Suppose that (X, λ) is a regular approach space. We will prove that the topological coreflection $(X, \mathcal{T}_\lambda)$ is regular. Therefore consider a filter $\mathcal{F} \in \mathbf{F}(X)$ such that $\mathcal{F} \to x$ in $(X, \lambda_\mathcal{T})$. By proposition 2.2.9 we have that $\lambda\mathcal{F}(x) = 0$. Since (X, λ) is a regular approach space, this gives us $\lambda\mathcal{F}^{(0)}(x) = 0$ or, again by the same proposition, $\mathcal{F}^{(0)} \to x$. Since the closure of a set A in the topological coreflection is given by $A^{(0)}$, we have that $\overline{\mathcal{F}} \to x$.

2. Suppose that $(X, \mathcal{T})$ is a regular topological space. We have to prove that $(X, \lambda_\mathcal{T})$ is regular. Therefore we have to check the inequality in proposition 4.3.2 1). We claim that this inequality only needs to be checked for $\gamma = \infty$ and $\gamma = 0$. Suppose that $\gamma < \infty$. Then we have $F^{(\gamma)} = \{x \in X \mid \delta_\mathcal{T}(x, F) \leq \epsilon\} = \{x \in X \mid \delta_\mathcal{T}(x, F) = 0\} = F^{(0)}$.
   Take now $\mathcal{F} \in \mathbf{F}(X)$ and $x \in X$. First suppose that $\gamma = \infty$. Then we immediately have $\lambda_\mathcal{T}\mathcal{F}^{(\gamma)}(x) \leq \lambda_\mathcal{T}\mathcal{F}(x) + \gamma$.
   Now suppose that $\gamma < \infty$. Then we have two possibilities: $\lambda_\mathcal{T}\mathcal{F}(x) = \infty$ and $\lambda_\mathcal{T}\mathcal{F}(x) = 0$. If $\lambda_\mathcal{T}\mathcal{F}(x) = \infty$, then the inequality is immediate. Consider therefore $\lambda_\mathcal{T}\mathcal{F}(x) = 0$. This means, by definition of the topological limit operator, that x is a topological limit point of the filter x. Since $(X, \mathcal{T})$ is regular, this gives us that x is also a topological limit point of the filter $\overline{\mathcal{F}}$. This implies that $\lambda_\mathcal{T}\mathcal{F}^{(0)} = 0$, which proves the inequality.

□

Notice that we did not prove that if an approach space has a regular topological coreflection, then the approach space is regular. In general this will not hold and a counterexample will be given later on in this section (example 4.3.9).

First of all we would like to give some characterizations of regularity for the limit operator using ultrafilters. Later on we will give an analogous characterization theorem for filters instead of ultrafilters.

Let X be an approach space, let $(t_\epsilon)_\epsilon$ be the tower on X and λ be the limit operator on X. By the transition theorems we know that $\mathcal{F} \xrightarrow{t_\epsilon} x$ if and only if $\lambda\mathcal{F}(x) \leq \epsilon$, for all $\mathcal{F} \in \mathbf{F}(X)$ and $x \in X$.

Theorem 4.3.5 *For an approach space X, the following are equivalent:*

1. *For $\mathcal{W},\mathcal{U} \in \boldsymbol{U}(X)$, $\gamma \in [0,\infty]$ and $x \in X$ we have*
$$\mathcal{W}^{(\gamma)} \subseteq \mathcal{U} \Rightarrow \lambda\mathcal{U}(x) \leq \lambda\mathcal{W}(x) + \gamma,$$

2. *For $\Theta \in \boldsymbol{U}^2(X),\mathcal{U} \in \boldsymbol{U}(X)$ and $x \in X$, we have*
$$\lambda\mathcal{U}(x) \leq \sup_{\mathcal{A}\in\Theta, B\in\mathcal{U}} \inf_{\mathcal{W}\in\mathcal{A}, b\in B} \lambda\mathcal{W}(b) + \lambda\Sigma\Theta(x),$$

3. *If J is a non-empty set, $\psi : J \to X, \sigma : J \to \boldsymbol{U}(X)$, and $\mathcal{U} \in \boldsymbol{U}(J)$, then*
$$\lambda\psi\mathcal{U} \leq \lambda\Sigma\sigma\mathcal{U} + \sup_{y\in J} \lambda\sigma(y)(\psi(y)),$$

4. *Let $\epsilon, \gamma \in [0,\infty]$, and let J be a non-empty set. Let $\psi : J \to X$ and $\sigma : J \to \boldsymbol{U}(X)$ such that $\sigma(y) \xrightarrow{t_\epsilon} \psi(y)$, for all $y \in J$. If $\mathcal{U} \in \boldsymbol{U}(J)$ is such that $\Sigma\sigma\mathcal{U} \xrightarrow{t_\gamma} x$, then $\psi(\mathcal{U}) \xrightarrow{t_{\epsilon+\gamma}} x$,*

5. *Let $\epsilon, \gamma \in [0,\infty]$, $\mathcal{U} \in \boldsymbol{U}(X)$. Then*
$$\mathcal{U} \xrightarrow{t_\epsilon} x \Rightarrow \mathcal{U}^{(\gamma)} \xrightarrow{t_{\epsilon+\gamma}} x.$$

Proof. 1. $\Rightarrow$ 2.: Suppose that condition 1. is fulfilled. Let $\Theta \in \mathbf{U}^2(X),\mathcal{U} \in \mathbf{U}(X)$ and $x \in X$. We put $\gamma = \lambda\Sigma\Theta(x)$ and $\epsilon = \sup_{\mathcal{A}\in\Theta,B\in\mathcal{U}} \inf_{\mathcal{W}\in\mathcal{A},b\in B} \lambda\mathcal{W}(b)$. It is sufficient to assume that both γ and ϵ are finite. Let $0 < \rho < \infty$ be arbitrary. Consider the subset
$$S = \big\{(\mathcal{G}, y) \mid \lambda\mathcal{G}(y) \leq \epsilon + \rho\big\} \subseteq \mathbf{U}(X) \times X.$$
The filterbasis $\Theta \times \mathcal{U}$ has a trace on S, so we can choose $\mathcal{R} \in \mathbf{U}(S)$ refining this trace. For $A \in \Sigma\Theta$ there exists $R_1 \in \mathcal{R}$ such that $A \in \bigcap_{z\in R_1} \pi_1 z$ with π_1 the projection on the first component restricted to S. This follows from the following implications:
$$\begin{aligned}
&A \in \Sigma\Theta\\
&\Rightarrow \tilde{A} \in \Theta\\
&\Rightarrow \{\mathcal{W} \in \mathbf{U}(X) \mid A \in \mathcal{W}\} \in \Theta\\
&\Rightarrow \exists R_1 \in \mathcal{R} : \pi_1 R_1 \subset \{\mathcal{W} \in \mathbf{U}(X) \mid A \in \mathcal{W}\}\\
&\Rightarrow \exists R_1 \in \mathcal{R}, \forall z \in R_1 : \pi_1 z \in \{\mathcal{W} \in \mathbf{U}(X) \mid A \in \mathcal{W}\}\\
&\Rightarrow \exists R_1 \in \mathcal{R} : \forall z \in R_1 : A \in \pi_1 z\\
&\Rightarrow \exists R_1 \in \mathcal{R} : A \in \textstyle\bigcap_{z\in R_1} \pi_1 z.
\end{aligned}$$
For $U \in \mathcal{U}$ there exists $R_2 \in \mathcal{R}$ such that $U = \pi_2 R_2$ with π_2 the projection on the second component restricted to S. For $z \in R_1 \cap R_2$, we have $\lambda\pi_1 z(\pi_2 z) \leq \epsilon + \rho$ and $\pi_2 z \in A^{(\epsilon+\rho)} \cap U$. Finally we can conclude that $(\Sigma\Theta)^{(\epsilon+\rho)} \subseteq \mathcal{U}$. This implies the following:
$$\begin{aligned}
\lambda\mathcal{U}(x) &\leq \lambda\Sigma\Theta(x) + \epsilon + \rho\\
&\leq \gamma + \epsilon + \rho.
\end{aligned}$$

By arbitrariness of ρ, this completes the first implication.

2. $\Rightarrow$ 3.: Suppose that J is a non-empty set, $\psi : J \to X, \sigma : J \to \mathbf{U}(X)$, and $\mathcal{U} \in \mathbf{U}(J)$. Take now $\Theta = \sigma\mathcal{U}$. By applying 2. we get

$$\lambda\psi\mathcal{U}(x) \leq \sup_{\mathcal{A}\in\Theta, B\in\psi\mathcal{U}} \inf_{\mathcal{W}\in\mathcal{A}, b\in B} \lambda\mathcal{W}(b) + \lambda\Sigma\sigma\mathcal{U}(x).$$

It remains to show that

$$\sup_{\mathcal{A}\in\sigma\mathcal{U}, B\in\psi\mathcal{U}} \inf_{\mathcal{W}\in\mathcal{A}, b\in B} \lambda\mathcal{W}(b) \leq \sup_{y\in J} \lambda\sigma(y)(\psi(y)).$$

We prove this as follows:

$$\begin{array}{rcl} \sup_{\mathcal{A}\in\sigma\mathcal{U}, B\in\psi\mathcal{U}} \inf_{\mathcal{W}\in\mathcal{A}, b\in B} \lambda\mathcal{W}(b) & = & \sup_{A\in\mathcal{U}, C\in\mathcal{U}} \inf_{\mathcal{W}\in\sigma(A), b\in\psi(C)} \lambda\mathcal{W}(b) \\ & = & \sup_{A\in\mathcal{U}, C\in\mathcal{U}} \inf_{a\in A, c\in C} \lambda\sigma(a)(\psi(c)) \\ & \leq & \inf_{a\in J, c\in J} \lambda\sigma(a)(\psi(c)) \\ & \leq & \sup_{y\in J} \lambda\sigma(y)(\psi(y)). \end{array}$$

This proves this implication.

3. $\Rightarrow$ 4.: Suppose that (X, λ) satisfies 3. Let $\epsilon, \gamma \in [0, \infty]$, and let J be a non-empty set. Let $\psi : J \to X$ and $\sigma : J \to \mathbf{U}(X)$ such that $\sigma(y) \xrightarrow{t_\epsilon} \psi(y)$, for all $y \in J$. Consider $\mathcal{U} \in \mathbf{U}(J)$ such that $\Sigma\sigma\mathcal{U} \xrightarrow{t_\gamma} x$. We have to prove that $\psi(\mathcal{U}) \xrightarrow{t_{\epsilon+\gamma}} x$. By the correspondence between the limit operator and the tower, we get $\lambda\sigma(y)(\psi(y)) \leq \epsilon$ and $\lambda\Sigma\sigma\mathcal{U}(x) \leq \gamma$. By applying 3., we get

$$\lambda\psi\mathcal{U}(x) \leq \lambda\Sigma\sigma\mathcal{U}(x) + \sup_{y\in J} \lambda\sigma(y)(\psi(y)) \leq \epsilon + \gamma.$$

This means precisely that $\psi(\mathcal{U}) \xrightarrow{t_{\epsilon+\gamma}} x$.

4. $\Rightarrow$ 5.: Assume that condition 4. is fulfilled and let $\mathcal{U} \xrightarrow{t_\epsilon} x$. For arbitrary $\gamma \in [0, \infty]$ let

$$J = \big\{(\mathcal{G}, y) \mid \mathcal{G} \in \mathbf{U}(X), y \in X, \text{ and } \mathcal{G} \xrightarrow{t_\gamma} y\big\}.$$

Define $\sigma : J \to \mathbf{U}(X)$ by $\sigma(\mathcal{G}, y) = \mathcal{G}$ and $\psi : J \to X$ by $\psi(\mathcal{G}, y) = y$. Then $\sigma(z) \xrightarrow{t_\gamma} \psi(z)$ holds for all $z \in J$. For each $U \in \mathcal{U}$, let $S_U = \{(\mathcal{G}, y) \in J \mid U \in \mathcal{G}\}$, and let $\mathcal{S}$ be the filter on J with basis $\{S_U : U \in \mathcal{U}\}$. We claim that $\mathcal{U} \subseteq \Sigma\sigma\mathcal{S}$. Take $U \in \mathcal{U}$. We need to prove that $U \in \Sigma\sigma\mathcal{S}$. By definition we have

$$\Sigma\sigma\mathcal{S} = \bigcup_{S\in\mathcal{S}} \bigcap_{z\in S} \sigma(z) = \bigcup_{G\in\mathcal{U}} \bigcap_{(\mathcal{G},y)\in S_G} \sigma(\mathcal{G}, y) = \bigcup_{G\in\mathcal{U}} \bigcap_{(\mathcal{G},y)\in S_G} \mathcal{G}.$$

Now take $U \in \mathcal{U}$. Then for all $(\mathcal{G}, y) \in S_U$ we have $U \in \mathcal{G}$. Hence, $U \in \Sigma\sigma\mathcal{S}$.

We need to prove that $\mathcal{U}^{(\gamma)} = \psi(\mathcal{S}) \xrightarrow{t_{\epsilon+\gamma}} x$. To show this we prove that every ultrafilter $\mathcal{W}$ finer that $\mathcal{U}^{(\gamma)}$ converges to x on the same level. Take $\mathcal{W}$ such an ultrafilter. We first claim that there exists an ultrafilter $\mathcal{A}$ on J which is finer than $\mathcal{S}$ such that $\psi(\mathcal{A}) \subseteq \mathcal{W}$. We prove this claim by contradiction. Suppose that for every ultrafilter $\mathcal{A}$ on J finer than $\mathcal{S}$ we have that $\psi(\mathcal{A}) \nsubseteq \mathcal{W}$. Then for every

ultrafilter $\mathcal{A} \in \mathbf{U}(\mathcal{S})$ there exists a set $A \in \mathcal{A}$ such that $\psi(A) \notin \mathcal{W}$. We denote this selection by $\rho(\mathcal{A})$. Then, by proposition 1.2.2, there exists a finite subset $U_\rho \subset \mathbf{U}(\mathcal{S})$ such that $\bigcup_{\mathcal{A}\in U_\rho} \rho(\mathcal{A}) \in \mathcal{S}$. But then we have that $\bigcup_{\mathcal{A}\in U_\rho} \psi(\rho(\mathcal{A})) = \psi(\bigcup_{\mathcal{A}\in U_\rho} \rho(\mathcal{A})) \in \psi(\mathcal{S})$ and since $\psi(\mathcal{S}) \subseteq \mathcal{W}$, we have $\bigcup_{\mathcal{A}\in U_\rho} \psi(\rho(\mathcal{A})) \in \mathcal{W}$. Since for all $\mathcal{A} \in U_\rho$ we had $\psi(\rho(\mathcal{A})) \notin \mathcal{W}$ and because of the fact that $\mathcal{W}$ is an ultrafilter, we find $X \setminus \psi(\rho(\mathcal{A})) \in \mathcal{W}$. Since U_ρ is finite, we also have $X \setminus \bigcup_{\mathcal{A}\in U_\rho} \psi(\rho(\mathcal{A})) = \bigcap_{\mathcal{A}\in U_\rho} X \setminus \psi(\rho(\mathcal{A})) \in \mathcal{W}$. This, however, is a contradiction. So take an ultrafilter $\mathcal{A}$ on J which is finer than $\mathcal{S}$ with the property that $\psi(\mathcal{A}) \subseteq \mathcal{W}$. Since $\mathcal{S} \subseteq \mathcal{A}$, we get $\Sigma\sigma\mathcal{S} \subseteq \Sigma\sigma\mathcal{A}$. We already showed that $\mathcal{U} \subseteq \Sigma\sigma\mathcal{S}$, hence $\mathcal{U} \subseteq \Sigma\sigma\mathcal{A}$. Since $\mathcal{U} \xrightarrow{t_\epsilon} x$, this implies $\Sigma\sigma\mathcal{A} \xrightarrow{t_\epsilon} x$. By applying 4., this gives us $\psi(\mathcal{A}) \xrightarrow{t_{\epsilon+\gamma}} x$. Since $\psi(\mathcal{A}) \subseteq \mathcal{W}$, we have $\mathcal{W} \xrightarrow{t_{\epsilon+\gamma}} x$. By the fact that we chose $\mathcal{W}$ arbitrary, we get $\mathcal{U}^{(\gamma)} \xrightarrow{t_{\epsilon+\gamma}} x$.

5. $\Rightarrow$ 1.: Suppose that condition 5. is fulfilled. Suppose $\mathcal{U}, \mathcal{W} \in \mathbf{U}(X)$ such that $\mathcal{W}^{(\gamma)} \subseteq \mathcal{U}$. Suppose that $\lambda\mathcal{W}(x) \leq \epsilon$. Then we have to prove that $\lambda\mathcal{U}(x) \leq \epsilon + \gamma$. By the correspondence between the limit operator and the tower, we get $\mathcal{W} \xrightarrow{t_\epsilon} x$. By applying 5. we get $\mathcal{W}^{(\gamma)} \xrightarrow{t_{\epsilon+\gamma}} x$. Again by the correspondence between the tower and the limit operator, this gives us $\lambda\mathcal{W}^{(\gamma)}(x) \leq \epsilon + \gamma$. Then we have $\lambda\mathcal{U}(x) \leq \lambda\mathcal{W}^{(\gamma)}(x) \leq \epsilon + \gamma$. This proves the last implication. □

In the foregoing theorem characterizations 1. and 2. were introduced by K. Robeys in [14], whereas 3. and 4. were introduced by P. Brock and D. Kent in [5].

We will now give an analogous theorem with characterizations of regularity using the limit operator, but now we will only require filters instead of ultrafilters.

Theorem 4.3.6 *For an approach space X the following are equivalent:*

1. *For $\mathcal{F} \in \boldsymbol{F}(X), x \in X$ and $\gamma \in [0, \infty]$ we have*
$$\lambda\mathcal{F}^{(\gamma)}(x) \leq \lambda\mathcal{F}(x) + \gamma,$$
2. *If J is a non-empty set, $\psi : J \to X, \sigma : J \to \boldsymbol{F}(X)$, and $\mathcal{F} \in \boldsymbol{F}(J)$, then*
$$\lambda\psi\mathcal{F} \leq \lambda\Sigma\sigma\mathcal{F} + \sup_{y\in J} \lambda\sigma(y)(\psi(y)),$$
3. *Let $\epsilon, \gamma \in [0, \infty]$, and let J be a non-empty set. Let $\psi : J \to X$ and $\sigma : J \to \boldsymbol{F}(X)$ such that $\sigma(y) \xrightarrow{t_\epsilon} \psi(y)$, for all $y \in J$. If $\mathcal{F} \in \boldsymbol{F}(J)$ is such that $\Sigma\sigma\mathcal{F} \xrightarrow{t_\gamma} x$, then $\psi\mathcal{F} \xrightarrow{t_{\epsilon+\gamma}} x$*
4. *Let $\epsilon, \gamma \in [0, \infty], \mathcal{F} \in \boldsymbol{F}(X)$. Then*
$$\mathcal{F} \xrightarrow{t_\epsilon} x \Rightarrow \mathcal{F}^{(\gamma)} \xrightarrow{t_{\epsilon+\gamma}} x.$$

Proof. 1. $\Leftrightarrow$ 4.: This follows immediately from the correspondence between the limit operator on X and the tower on X.

4. $\Rightarrow$ 3.: Let J, σ, ψ and $\mathcal{F} \in \mathbf{F}(J)$ be as in 3., and assume $\Sigma\sigma\mathcal{F} \xrightarrow{t_\gamma} x$. We know that $\sigma(y) \xrightarrow{t_\epsilon} \psi(y)$, for all $y \in J$. We claim now that this gives us that $(\Sigma\sigma\mathcal{F})^{(\epsilon)} \subseteq \psi\mathcal{F}$. Since $\sigma(y) \xrightarrow{t_\epsilon} \psi(y)$, for all $y \in J$, we get that for all $F \in \sigma(y)$ we have that $\psi(y) \in F^{(\epsilon)}$. Hence $\psi(y) \in \bigcap_{F\in\sigma(y)} F^{(\epsilon)}$. Now take $F \in (\Sigma\sigma\mathcal{F})^{(\epsilon)}$. Then we have the following:

$$\begin{aligned}
&F \in (\Sigma\sigma\mathcal{F})^{(\epsilon)}\\
&\Rightarrow \exists A \in \Sigma\sigma\mathcal{F} : A^{(\epsilon)} \subset F\\
&\Rightarrow \exists G \in \mathcal{F}\, \forall z \in G : A \in \sigma(z) \text{ and } A^{(\epsilon)} \subset F\\
&\Rightarrow \exists G \in \mathcal{F}\, \forall z \in G : \psi(z) \in A^{(\epsilon)} \text{ and } A^{(\epsilon)} \subset F\\
&\Rightarrow \exists G \in \mathcal{F} : \psi(G) \subset A^{(\epsilon)} \subset F\\
&\Rightarrow \exists G \in \mathcal{F} : \psi(G) \subset F\\
&\Rightarrow F \in \psi(\mathcal{F}).
\end{aligned}$$

But $(\Sigma\sigma\mathcal{F})^{(\epsilon)} \xrightarrow{t_{\epsilon+\gamma}} x$ by hypothesis, and consequently $\psi\mathcal{F} \xrightarrow{t_{\epsilon+\gamma}} x$, establishing 3.

3. $\Rightarrow$ 4.: This proof is analogous to the proof of 4. $\Rightarrow$ 5. in theorem 4.3.5.

3. $\Rightarrow$ 2.: Let J, ψ, σ and $\mathcal{F} \in \mathbf{F}(J)$ be as in 2. Suppose that $\sup_{y\in J} \lambda\sigma(y)(\psi(y)) \leq \epsilon$. This means that for every $y \in J$ we have that $\lambda\sigma(y)(\psi(y)) \leq \epsilon$. Using the correspondence between the limit operator and the tower on X, this gives us $\sigma(y) \xrightarrow{t_\epsilon} \psi(y)$, for all $y \in Y$. If we suppose $\lambda\Sigma\sigma\mathcal{F} \leq \gamma$, the same correspondence gives us $\Sigma\sigma\mathcal{F} \xrightarrow{t_\gamma} x$. By applying 3., we get $\psi\mathcal{F} \xrightarrow{t_{\epsilon+\gamma}} x$, or, again by the same correspondence, $\lambda\psi\mathcal{F}(x) \leq \epsilon + \gamma$.

2. $\Rightarrow$ 3.: This proof is analogous to the proof of 3. $\Rightarrow$ 4. in theorem 4.3.5. $\square$

The regularity condition, as stated above, is of "purely approach" nature. We will give an example of an approach space which is not regular, but where its topological coreflection is regular. Before we are able to do this, we need to prove some propositions.

The following proposition gives a characterization of regularity using the gauge in an approach space where the gauge has a symmetric basis.

Proposition 4.3.7 *If the approach space X has a basis for the gauge consisting of metrics, then the approach space is regular.*

Proof. Let X be an approach space and let $\mathcal{H}$ be a symmetric gauge basis. Let $\mathcal{F}$ be a filter on X and let $\epsilon > 0$. Then for any $F \in \mathcal{F}, z \in F^{(\epsilon)}, d \in \mathcal{H}$ and $\epsilon' > \epsilon$ there exists $y_d^z \in F$ such that $d(z, y_d^z) \leq \epsilon'$. Hence for any $x \in X$ we have

$$\begin{aligned}
\lambda\mathcal{F}^{(\epsilon)}(x) &= \sup_{d\in\mathcal{H}} \inf_{F\in\mathcal{F}} \sup_{z\in F^{(\epsilon)}} d(x,z)\\
&\leq \sup_{d\in\mathcal{H}} \inf_{F\in\mathcal{F}} \sup_{z\in F^{(\epsilon)}} \inf_{y\in F} \big(d(x,y) + d(y,z)\big)\\
&\leq \sup_{d\in\mathcal{H}} \inf_{F\in\mathcal{F}} \sup_{z\in F^{(\epsilon)}} \big(d(x,y_d^z) + d(y_d^z,z)\big)\\
&\leq \sup_{d\in\mathcal{H}} \inf_{F\in\mathcal{F}} \sup_{z\in F^{(\epsilon)}} d(x,y_d^z) + \epsilon'\\
&\leq \sup_{d\in\mathcal{H}} \inf_{F\in\mathcal{F}} \sup_{y\in F} d(x,y) + \epsilon'\\
&= \lambda\mathcal{F}(x) + \epsilon'
\end{aligned}$$

which by arbitrariness of x and $\epsilon' > \epsilon$ shows that $\lambda\mathcal{F}^{(\epsilon)} \leq \lambda\mathcal{F} + \epsilon$. $\square$

The following proposition gives a characterization of regularity using the gauge in quasi-metric approach spaces.

Proposition 4.3.8 *A quasi-metric approach space (X, δ_d) is regular if and only if d is a metric.*

Proof. The if-part follows from proposition 4.3.7, since for metric spaces there exists a basis of the gauge $\mathcal{G}$ consisting of metrics.

To show the only-if-part let $x, y \in X$ and let $\epsilon \in [0, \infty]$. Then

$$\lambda \dot{x}(y) = d(y, x)$$

and

$$\lambda(\dot{x})^{(\epsilon)}(y) = \sup_{z \in \{x\}^{(\epsilon)}} d(y, z).$$

Hence it follows from the definition of regularity that $\sup_{z \in \{x\}^{(\epsilon)}} d(y, z) \leq d(y, x) + \epsilon$ and thus

$$d(z, x) \leq \epsilon \Rightarrow d(y, z) \leq d(y, x) + \epsilon.$$

If we take $y = x$, then we get

$$d(z, x) \leq \epsilon \Rightarrow d(x, z) \leq \epsilon$$

which by arbitrariness of x, z and ϵ shows that $d(z, x) = d(x, z)$. □

We are now able to construct an example which shows that there exist approach spaces which are not regular, but which do have a regular topological coreflection.

Example 4.3.9

Consider the real line equipped with the quasi-metric

$$d(x, y) := \begin{cases} |x - y| & x \leq y \\ 2|x - y| & y \leq x. \end{cases}$$

$(\mathbb{R}, \delta_d)$ is not regular by proposition 4.3.8. The topological coreflection of this approach space is the usual Euclidean topology.

Up until now we have given various characterizations of regularity. We have characterizations using the limit operator λ of the approach space and one using the tower $(t_\epsilon)_\epsilon$. In the following theorem we give a characterization using the gauge $\mathcal{G}$ of the approach space.

Theorem 4.3.10 *For an approach space (X, δ) the following are equivalent:*

1. *(X, δ) is regular in the sense of proposition 4.3.2:*

$$\forall \mathcal{F} \in \boldsymbol{F}(X), \forall \epsilon \in [0, \infty] : \lambda \mathcal{F}^{(\epsilon)} \leq \lambda \mathcal{F} + \epsilon$$

2. $\forall x \in X, \forall \epsilon_2 > \epsilon_1 > 0, \forall \theta > 0, \forall p \in \mathcal{G}, \exists q \in \mathcal{G}$ *such that*

$$\{q(x,\cdot) < \epsilon_1\}^{(\theta)} \subseteq \{p(x,\cdot) < \epsilon_2 + \theta\},$$

3. $\forall x \in X, \forall A \in 2^X, \forall \epsilon, \theta > 0 :$

$$\{p(x,\cdot) < \epsilon\}^{(\theta)} \cap A \neq \emptyset, \forall p \in \mathcal{G} \Rightarrow \delta(x,A) \leq \epsilon + \theta,$$

4. $\forall x \in X, \forall A \in 2^X, \forall \epsilon, \theta > 0 : \delta(x,A) > \epsilon + \theta \Rightarrow \exists p \in \mathcal{G}, \exists (q^a)_{a\in A} \in \mathcal{G}$ *such that* $\{p(x,\cdot) < \epsilon\} \cap \{\inf_{a\in A} q^a(a,\cdot) < \theta\} = \emptyset$.

Proof. First we show 1. implies 2. If $x \in X$ and $\epsilon_1 > 0$, consider the filter $\mathcal{F}$ generated by the family $\{p(x,\cdot) < \epsilon_1 \mid p \in \mathcal{G}\}$. Then $\lambda\mathcal{F}(x) \leq \epsilon_1$. If $\epsilon_2 > \epsilon_1$ and $\theta > 0$ then we have by hypothesis

$$\lambda\mathcal{F}^{(\theta)}(x) \leq \lambda\mathcal{F}(x) + \theta \leq \epsilon_1 + \theta < \epsilon_2 + \theta.$$

So if $p \in \mathcal{G}$, then $\{p(x,\cdot) < \epsilon_2 + \theta\}$ contains a set $\{q(x,\cdot) < \epsilon_1\}^{(\theta)}$, with $q \in \mathcal{G}$.

That 2. implies 3. follows from the fact that

$$\delta(x,A) = \sup_{p\in\mathcal{G}} \inf_{a\in A} p(x,a).$$

Now let $x \in X, A \in 2^X$ and $\epsilon, \theta > 0$. If 3. holds and $\delta(x,A) > \epsilon + \theta$ then we can find $p \in \mathcal{G}$ such that $\{p(x,\cdot) < \epsilon\}^{(\theta)} \cap A = \emptyset$. For each $a \in A$ we now select $q^a \in \mathcal{G}$ such that

$$\{p(x,\cdot) < \epsilon\} \cap \{q^a(a,\cdot) < \theta\} = \emptyset.$$

Then $\inf_{a\in A} q^a \in \mathcal{G}$ and

$$\left\{\inf_{a\in A} q^a(a,\cdot) < \theta\right\} \cap \{p(x,\cdot) < \epsilon\} = \emptyset.$$

Conversely, if 4. holds and $\{p(x,\cdot) < \epsilon\}^{(\theta)} \cap A \neq \emptyset$ for each $p \in \mathcal{G}$, then we have that $\delta(x,A) = \sup_{p\in\mathcal{G}} \inf_{a\in A} p(x,a) \leq \epsilon + \theta$. Hence 3. and 4. are equivalent.

Finally we prove 3. implies 1. If $\mathcal{F} \in \mathbf{F}(X), \epsilon > 0$ and $x \in X$ are such that $\lambda\mathcal{F}(x) < \epsilon$ then for each $\theta > 0$ and $p \in \mathcal{G}$ we have $\{p(x,\cdot) < \epsilon\}^{(\theta)} \in \mathcal{F}^{(\theta)}$. Hence if $A \in \sec(\mathcal{F}^{(\theta)})$ then we have $\delta(x,A) \leq \epsilon + \theta$, by 3. So $\lambda\mathcal{F}^{(\theta)} = \sup_{A\in\sec(\mathcal{F}^{(\theta)})} \delta(x,A) \leq \epsilon + \theta$. □

We now define **RAP**, the construct of regular approach spaces and contractions. It is clear that the category **RAP** is a full and concrete subconstruct of **AP**. The following theorem gives an embedding result.

Theorem 4.3.11 *The category* ***RAP*** *is reflectively embedded in* ***AP****.*

Proof. In Adámek *et. al.* [1] it is shown that a concrete construct $(\mathcal{A}, U)$ is embedded as a concretely reflective subconstruct of a topological construct $(\mathcal{C}, V)$ if and only if $(\mathcal{A}, U)$ is initially closed in $(\mathcal{C}, V)$.

To this end, we consider an initial source

$$\big(f_j : (Y, \lambda_Y) \to (X_j, \lambda_j)\big)_{j \in J},$$

where $(Y, \lambda_Y) \in |\mathbf{AP}|$ and $(X_j, \lambda_j) \in |\mathbf{RAP}|$, for every $j \in J$. We have to show that (Y, λ_Y) is also a regular approach space.

Consider therefore $\mathcal{F} \in \mathbf{F}(Y)$ and $\gamma \in [0, \infty]$. First of all we notice that every function f_j, for $j \in J$ is a contraction, since the source is initial. Theorem 1.6.2 then gives us that

$$f_j\big(F^{(\gamma)}\big) \subseteq f_j(F)^{(\gamma)} \quad \forall j \in J.$$

Eventually this gives us the following relation between filters:

$$f_j(\mathcal{F})^{(\gamma)} \subseteq f_j\big(\mathcal{F}^{(\gamma)}\big).$$

By theorem 1.7.1 and the regularity of the approach spaces (X_j, λ_j) for $j \in J$, we have the following equalities:

$$\begin{aligned}
\lambda_Y \mathcal{F}^{(\gamma)} &= \sup_{j \in J} \lambda_j\big(f_j(\mathcal{F}^{(\gamma)})\big) \circ f_j \\
&\leq \sup_{j \in J} \lambda_j\big(f_j(\mathcal{F})^{(\gamma)}\big) \circ f_j \\
&\leq \sup_{j \in J} \lambda_j\big(f_j(\mathcal{F})\big) \circ f_j + \gamma \\
&= \lambda_Y \mathcal{F} + \gamma.
\end{aligned}$$

By arbitrariness of $\mathcal{F} \in \mathbf{F}(Y)$ and $\gamma \in [0, \infty]$, this shows that (Y, λ_Y) is regular. $\square$

We end this section about regularity with an application of regularity in an extension theorem. To this end we first give a well-known extension theorem in topological spaces, as stated in Bourbaki [3].

Let X be a topological space, A a subset of X and $f : A \to Y$ a mapping of A into a regular space Y. Consider the subset $X_0 = \{a \in \overline{A} \mid \lim f(\mathcal{V}(x)_{|A}) \text{ exists}\}$. We can extend $f : A \to Y$ to a continuous map $\tilde{f} : X_0 \to Y$ such that $\tilde{f}_{|A} = f$. For $a \in X_0 \setminus A$ we therefore define

$$\tilde{f}(a) := \lim f(\mathcal{V}(x)_{|A}).$$

We will now extend this theorem to the realm of approach theory. This theorem was first proved by Jäger in [8].

We consider the following situation. We have a mapping $f : A \to Y$, where $A \subseteq X$ and (X, λ_X) and (Y, λ_Y) are two approach spaces. If $f : (A, (\lambda_X)_A) \to (Y, \lambda_Y)$ is a contraction, we want to find the conditions under which we can extend f to a contraction g from $\overline{A}$ or a suitable subset of $\overline{A}$ to Y such that $g|_A = f$.

We want to point out that $A^{(0)}$ is the closure of A in the topological coreflection. Therefore we will often write $\overline{A}$ instead of $A^{(0)}$.

For $x \in X$ and $\epsilon \in [0, \infty]$ we introduce the following notation:

$$H_A^\epsilon(x) = \{\mathcal{F} \in \mathbf{F}(X) \mid A \in \mathcal{F}, \lambda_X(\mathcal{F})(x) \leq \epsilon\},$$

$$F_A^\epsilon(x) = \begin{cases} \{y \in Y \mid \lambda_Y(f(\mathcal{F}_{|A}))(y) \leq \epsilon \quad \forall \mathcal{F} \in H_A^\epsilon(x)\} & \text{if } H_A^\epsilon(x) \neq \emptyset, \\ Y & \text{if } H_A^\epsilon(x) = \emptyset. \end{cases}$$

We immediately see that for $\epsilon \leq \delta$, we have $H_A^\epsilon(x) \subseteq H_A^\delta(x)$. Next we claim that $x \in A^{(\epsilon)}$ if and only if $H_A^\epsilon(x) \neq \emptyset$. To prove this, we first consider $x \in A^{(\epsilon)}$. Then we get the existence of an ultrafilter $\mathcal{U}$ on X such that $A \in \mathcal{U}$ and $\lambda_X\mathcal{U}(x) \leq \epsilon$. Then $\mathcal{U} \in H_A^\epsilon(x)$ and hence $H_A^\epsilon(x) \neq \emptyset$. Conversely, suppose $H_A^\epsilon(x) \neq \emptyset$, then there exists a filter $\mathcal{F}$ on X such that $A \in \mathcal{F}$ and $\lambda_X\mathcal{F}(x) \leq \epsilon$. Now take an ultrafilter $\mathcal{U}$ on X, finer than $\mathcal{F}$. Then we have $\lambda_X\mathcal{U}(x) \leq \lambda_X\mathcal{F}(x) \leq \epsilon$ and $A \in \mathcal{U}$. Hence, $x \in A^{(\epsilon)}$.

Hence for a subset $A \subseteq X$ we have $\overline{A} = X$ if and only if, for all $x \in X$, $H_A^{(0)}(x) \neq \emptyset$.

For a subset $A \subseteq X$, we now define X_0 as follows

$$X_0 = \Big\{x \in \overline{A} \mid \bigcap_{\alpha \in [0,\infty]} F_A^\alpha(x) \neq \emptyset\Big\}.$$

The following lemma shows that this definition of X_0 makes sense.

Lemma 4.3.12 *Let $(X, \lambda_X), (Y, \lambda_Y)$ be approach spaces, $A \subseteq X$ and let $f : (A, (\lambda_X)_A) \to (Y, \lambda_Y)$ be a contraction. Then $A \subseteq X_0$.*

Proof. Let $x \in A$. Since $A \in \dot{x}$, $\dot{x} \in \mathbf{U}(X)$ and $\lambda(\dot{x})(x) = 0$, we have that $x \in A^{(0)}$. Hence $H_A^\epsilon(x) \neq \emptyset$, for all $\epsilon \in [0, \infty]$. Let now $\mathcal{F} \in H_A^\alpha(x)$ for some $\alpha \in [0, \infty]$. Then $A \in \mathcal{F}$ and $\lambda\mathcal{F}(x) \leq \alpha$. Hence $(\lambda_X)_A(\mathcal{F}_{|A})(x) \leq \alpha$. Since f is a contraction, we get $\lambda_Y(f(\mathcal{F}_{|A}))(f(x)) \leq \alpha$, i.e. $f(x) \in F_A^\alpha(x)$. Hence $f(x) \in \bigcap_{\alpha \in [0,\infty]} F_A^\alpha(x)$. □

We will now formulate and prove the extension theorem in **AP**

Theorem 4.3.13 *Let $(X, \lambda_X), (Y, \lambda_Y)$ be approach spaces and let (Y, λ_Y) be regular. Let $A \subseteq X$ and let $f : (A, (\lambda_X)_A) \to (Y, \lambda_Y)$ be a contraction. Then there exists a contraction $g : (X_0, (\lambda_X)_{X_0}) \to (Y, \lambda_Y)$ such that $g|_A = f$.*

Proof. For $x \in X_0 \setminus A$ we choose a value $y_x \in \bigcap_{\alpha \in [0,\infty]} F_A^\alpha(x)$ and define

$$g(x) := \begin{cases} f(x) & \text{If } x \in A, \\ y_x & \text{If } x \in X_0 \setminus A. \end{cases}$$

We show that g is a contraction. Let $\mathcal{G} \in \mathbf{F}(X_0)$, $x_0 \in X_0$ and let $\alpha = \lambda_X\mathcal{G}(x_0)$. We have to show that $\lambda_Y g(\mathcal{G})(g(x_0)) \leq \alpha$.

Consider the selection $\sigma : X_0 \to \mathbf{F}(X_0)$, where $\sigma(x) = \mathcal{F}_x$ such that $A \in \mathcal{F}_x$ and $\lambda\mathcal{F}_x(x) = 0$. The diagonal condition (L4) gives us

$$\begin{aligned} \lambda_X \Sigma\sigma\mathcal{G}(x_0) &\leq \lambda_X\mathcal{G}(x_0) + \sup_{x \in X_0} \lambda\mathcal{F}_x(x) \\ &= \lambda_X\mathcal{G}(x_0) \\ &= \alpha. \end{aligned}$$

Since $\Sigma\sigma\mathcal{G} = \bigcup_{G\in\mathcal{G}}\bigcap_{x\in G}\mathcal{F}_x$ and $A \in \mathcal{F}_x$, for every $x \in X_0$, we get $A \in \Sigma\sigma\mathcal{G}$. Hence $\Sigma\sigma\mathcal{G} \in H_A^\alpha(x)$ and since $\Sigma\sigma\mathcal{G}$ has a trace on A, we find

$$\lambda_Y f\big((\Sigma\sigma\mathcal{G})_{|A}\big)(g(x_0)) \leq \alpha.$$

We now consider the functions $\psi : X_0 \to Y$, where $\psi = g$ and $\rho : X_0 \to \mathbf{F}(Y)$, where $\rho(x) = f(\mathcal{F}_{x_{|A}})$. Y is regular and by using characterization 2. in theorem 4.3.6. we find

$$\begin{aligned}\lambda_Y g(\mathcal{G})(g(x_0)) &\leq \lambda_Y \Sigma\rho\mathcal{G}(g(x_0)) + \sup_{x\in X_0} \lambda_Y f(\mathcal{F}_{x_{|A}})(g(x)) \\ &= \lambda_Y \Sigma\rho\mathcal{G}(g(x_0)).\end{aligned}$$

We now have the following equalities:

$$\begin{aligned}\Sigma\rho\mathcal{G} &= \bigcup_{G\in\mathcal{G}}\bigcap_{x\in G} f(\mathcal{F}_{x_{|A}}) \\ &= f\Big(\bigcup_{G\in\mathcal{G}}\bigcap_{x\in G}\mathcal{F}_{x|A}\Big) \\ &= f\big(\Sigma\sigma\mathcal{G}_{|A}\big).\end{aligned}$$

From this we can conclude the following

$$\lambda_Y g(\mathcal{G})(g(x_0)) \leq \lambda_Y f\big(\Sigma\sigma\mathcal{G}_{|A}\big)(g(x_0)) \leq \alpha,$$

which completes the proof. □

Definition 4.3.14 *An approach space (X, λ) is called a **T2-space** or **Hausdorff** if $x = y$ whenever $\lambda\mathcal{F}(x) = 0$ and $\lambda\mathcal{F}(y) = 0$ for some filter $\mathcal{F}$ on X.*

Hence an approach space is Hausdorff if and only if its topological coreflection is Hausdorff. For more information on separation in **AP**, we refer to the article 'A Note on Separation in **AP**' by R. Lowen and M. Sioen [11].

If (Y, λ_Y) is a T2-space then we see that for each $x \in X$, $F_A^0(x)$ has at most one point. In this case $\bigcap_{\alpha\in[0,\infty]} F_A^\alpha(x)$ has at most one point and the extension g will be unique.

Theorem 4.3.15 *Let $(X, \lambda_X), (Y, \lambda_Y)$ be approach spaces and let (Y, λ_Y) be a regular T2-space. Let further $A \subseteq X$ be such that $A^{(0)} = X$ and let $f : (A, (\lambda_X)_A) \to (Y, \lambda_Y)$ be a contraction. The following are equivalent:*
(1) There is a unique contraction $g : (X, \lambda_X) \to (Y, \lambda_Y)$ such that $g|_A = f$.
(2) For each $x \in X$, $\bigcap_{\alpha\in[0,\infty]} F_A^\alpha(x) \neq \emptyset$.

Proof. If f has such an extending contraction $g : (X, \lambda_X) \to (Y, \lambda_Y)$, then for $x \in X$ we have $g(x) \in \bigcap_{\alpha\in[0,\infty]} F_A^\alpha(x)$. Indeed, because $A^{(0)} = X$ we know that all $H_A^\alpha(x) \neq \emptyset$. For $\mathcal{F} \in H_A^\alpha(x)$ we know that $A \in \mathcal{F}$ and $\lambda_X(\mathcal{F})(x) \leq \alpha$. Hence

$$\lambda_Y g(\mathcal{F})(g(x)) \leq \alpha.$$

We conclude $\lambda_Y f(\mathcal{F}_{|A})(g(x)) = \lambda_Y g(\mathcal{F}_{|A})(g(x)) \leq \lambda_Y g(\mathcal{F})(g(x)) \leq \alpha$, i.e. $g(x) \in F_A^\alpha(x)$.

If, conversely, $\bigcap_{\alpha \in [0,\infty]} F_A^\alpha(x) \neq \emptyset$ for all $x \in X$, then because $X_0 = A^{(0)} = X$ we have a unique contraction $g : (X, \lambda_X) \to (Y, \lambda_Y)$ with $g|_A = f$, by theorem 4.3.13. $\square$

Chapter 5

AN ASCOLI THEOREM IN APPROACH THEORY

In this last chapter we will establish an application of approach theory to function spaces. The goal of this chapter is to prove a general Ascoli theorem in the setting of approach theory. The theorem of Ascoli describes (pre)compact subsets of function spaces. The classical theorem of Ascoli as we can find in Bourbaki [4], states the following. Let X, Y be uniform spaces, let Σ be an open cover of X which is closed under finite unions and let $\mathcal{H}$ be any collection of functions from X to Y. If each set in Σ is precompact, if for each $A \in \Sigma$ the collection of $\mathcal{H}_{|A}$ is uniformly equicontinuous and if for each $x \in X$, $\mathrm{ev}_x(\mathcal{H})$ is precomapct, then $\mathcal{H}$ is precompact.

It is clear that, before we can prove this theorem, we need to extend the setting of uniform spaces to approach theory. Therefore, we start chapter 5 with a study of uniform gauge spaces. We will also need a quantified version of precompactness in approach theory. To this end, we will construct the measure of precompactness. We will compare this definition with the definition of precompactness in uniform spaces and give some good consistency results. Finally, we will need a notion of equicontractivity. We will define when a set $\mathcal{H} \subset Y^X$ is uniformly equicontractive and link this to the uniform concept of uniform equicontinuity. We will also introduce a measure of uniform contractivity and uniform equicontractivity.

This all will enable us to prove a general Ascoli theorem in approach theory. We will see that this theorem, just as many other theorems on the approach level, appears to have no conditions, meaning that the conditions are encapsulated in the inequality formulating the theorem.

For this chapter we refer to the paper 'An Ascoli theorem in approach theory' by R. Lowen [10].

5.1 Uniform gauge spaces

Before we will be able to start our exposition about Ascoli's theorem, we need to learn more about uniform gauge spaces. This section is fully dedicated to this subject.

Given a set X, a collection $\mathcal{G} \subset p\mathrm{M}^\infty(X)$ and an ∞p-metric d, we will say

that d is **uniformly dominated** by $\mathcal{G}$ or that $\mathcal{G}$ dominates d uniformly, if for all $\epsilon > 0$ and $\omega < \infty$ there exists a $d^{\epsilon,\omega} \in \mathcal{G}$ such that

$$d \wedge \omega \leq d^{\epsilon+\omega} + \epsilon.$$

We will then also say that the family $(d^{\epsilon,\omega})_{\epsilon>0,\omega<\infty}$ dominates d uniformly. Further we will say that a collection of ∞p-metrics $\mathcal{G}$ is **uniformly saturated**, if any ∞p-metric d which is uniformly dominated by $\mathcal{G}$ already belongs to $\mathcal{G}$.

Definition 5.1.1 *A subset $\mathcal{G}$ of $pM^\infty(X)$ is called a **uniform gauge** if it is an ideal in $pM^\infty(X)$ which fulfills the following property:*

(UG1) $\mathcal{G}$ is uniformly saturated.

As was the case for approach spaces, here too it regularly happens that one has a collection of ∞p-metrics which would be a natural candidate to form a uniform gauge but not all conditions are fulfilled. The following type of collection will often be encountered.

Definition 5.1.2 *A subset $\mathcal{H}$ of $pM^\infty(X)$ is called a **uniform gauge basis** if it is an ideal in $pM^\infty(X)$.*

By definition, a uniform gauge is also a uniform gauge basis, and similarly to the situation for approach spaces, here too, any result shown to hold for uniform gauge bases will also hold for uniform gauges.

In order to derive the uniform gauge from a uniform gauge basis we will require a **uniform saturation operation** which is similar to the saturation operation for gauges. Given a subset $\mathcal{D} \subset p\mathrm{M}^\infty(X)$ we define

$$\widehat{\mathcal{D}} := \{d \in p\mathrm{M}^\infty(X) \mid \mathcal{D} \text{ dominates } d \text{ uniformly}\}.$$

We call $\widehat{\mathcal{D}}$ the **uniform saturation** of $\mathcal{D}$.

Definition 5.1.3 *An ideal basis $\mathcal{H}$ in $pM^\infty(X)$ is said to be a **basis for a uniform gauge** if $\widehat{\mathcal{H}} = \mathcal{G}$. In this case we say that $\mathcal{H}$ generates $\mathcal{G}$ uniformly and that $\mathcal{G}$ is uniformly generated by $\mathcal{H}$.*

Theorem 5.1.4 *If $\mathcal{H}$ is a uniform gauge basis, then $\widehat{\mathcal{H}}$ is a uniform gauge with $\mathcal{H}$ as basis and if $\mathcal{H}$ is a basis for a uniform gauge $\mathcal{G}$, then it is a uniform gauge basis.*

Proof. This follows immediately from the definitions. □

Definition 5.1.5 *A pair $(X, \mathcal{G})$ where $\mathcal{G}$ is a uniform gauge on X is called a **uniform gauge space** or shortly, a UG-space.*

The associated morphisms are defined in the same way as in the approach case.

Definition 5.1.6 *Let $(X, \mathcal{G})$ and $(X', \mathcal{G}')$ be uniform gauge spaces and let $f : X \to X'$ be a function. We say that f is a* ***uniform contraction*** *if*

$$\forall d \in \mathcal{G}' : d \circ (f \times f) \in \mathcal{G}.$$

From the uniform saturation condition we immediately find that this is equivalent to the statement that

$$\forall d \in \mathcal{G}' \, \forall \epsilon > 0 \, \forall \omega < \infty \, \exists e \in \mathcal{G} : d \circ (f \times f) \wedge \omega \leq e + \epsilon.$$

Uniform gauge spaces and uniform contractions form a category which we denote **UG**.

Theorem 5.1.7 ***UG*** *is a topological category.*

Proof. Given uniform gauge spaces $(X_j)_{j \in J}$, consider the source

$$(f_j : X \to X_j)_{j \in J}$$

in **UG**. If for each $j \in J$, $\mathcal{H}_j$ is a basis for the uniform gauge on X_j, then a basis for the initial uniform gauge on X is given by

$$\mathcal{H} := \Big\{ \sup_{j \in K} d_j \circ (f_j \times f_j) \mid K \in 2^{(J)}, \forall j \in K : d_j \in \mathcal{H}_j \Big\}.$$

By definition of $\mathcal{H}$, all functions f_j, for $j \in J$, become uniform contractions.

Consider now a uniform gauge space $(Y, \mathcal{G}_Y)$ and a function $f : Y \to X$. First we suppose that f is a uniform contraction. For $d_j \in \mathcal{H}_j$ we get

$$d_j \circ \big((f_j \circ f) \times (f_j \circ f)\big) = d_j \circ (f_j \times f_j) \circ (f \times f).$$

Since f_j is a uniform contraction, we know that $d_j \times (f_j \times f_j) \in \mathcal{H}$. If we now use the fact that f is a uniform contraction, this gives us $d_j \circ \big((f_j \circ f) \times (f_j \circ f)\big) \in \mathcal{G}_Y$. Hence, $f_j \circ f$ is a uniform contraction for every $j \in J$. Now suppose that the functions $f_j \circ f$ are uniform contractions, for all $j \in J$. Consider $d' \in \widehat{\mathcal{H}}$. Choose $\epsilon > 0$ and $\omega < \infty$. Then there exists a finite subset $K \subset J$ and for all $j \in K$ there exists $d_j \in \mathcal{H}_j$ such that

$$d' \circ (f \times f) \wedge \omega \leq \sup_{j \in K} d_j \circ (f_j \times f_j) \circ (f \times f) + \epsilon = \sup_{j \in K} d_j \circ \big((f_j \circ f) \times (f_j \circ f)\big) + \epsilon.$$

Since $f_j \circ f$ is a uniform contraction, for every $j \in J$, we get $\sup_{j \in K} d_j \circ \big((f_j \circ f) \times (f_j \circ f)\big) \in \mathcal{G}_Y$. By arbitrariness of ω and ϵ this gives us $d' \circ (f \times f) \in \mathcal{G}_Y$, hence f is a uniform contraction. □

We will now describe the embedding of the category **Unif** of uniform spaces and uniformly continuous functions into the category **UG**.

Theorem 5.1.8 *Suppose $(X, \mathcal{U})$ is a uniform space. Then the collection $\mathcal{G}_{\mathcal{U}}$ of all uniformly continuous ∞p-metrics is a uniform gauge.*

Proof. The collection $\mathcal{G}_{\mathcal{U}}$ of all uniformly continuous ∞p-metrics for $\mathcal{U}$ is an ideal which satisfies the condition that if e is an ∞p-metric and

$$\forall \epsilon > 0 \exists \delta > 0 \exists d \in \mathcal{G}_{\mathcal{U}} : \{d < \delta\} \subset \{e < \epsilon\}$$

then $e \in \mathcal{G}_{\mathcal{U}}$. Clearly this implies that $\mathcal{G}_{\mathcal{U}}$ is uniformly saturated. □

Definition 5.1.9 *A uniform gauge space of type $(X, \mathcal{G}_{\mathcal{U}})$ for some uniformity $\mathcal{U}$ on X, will be called a* ***uniform uniform gauge space*** *or, somewhat less unfortunate, a uniform UG-space.*

The fact that **Unif** is concretely embedded in **UG** is an immediate consequence of the following proposition.

Proposition 5.1.10 *Let $(X,\mathcal{U})$ and $(X',\mathcal{U}')$ be uniform spaces and consider the function $f : X \to X'$. Then the following are equivalent:*

1. *$f : (X,\mathcal{U}) \to (X',\mathcal{U}')$ is uniformly continuous,*
2. *$f : (X, \mathcal{G}_{\mathcal{U}}) \to (X', \mathcal{G}_{\mathcal{U}'})$ is a uniform contraction.*

Proof. Suppose that the first condition is fulfilled. For any uniformly continuous ∞p-metric on X', we find that $d \circ (f \times f)$ is a uniformly continuous ∞p-metric on X. Hence, for any $d \in \mathcal{G}_{\mathcal{U}'}$, we have that $d \circ (f \times f) \in \mathcal{G}_{\mathcal{U}}$. Hence, $f : (X, \mathcal{G}_{\mathcal{U}}) \to (X', \mathcal{G}_{\mathcal{U}'})$ is a uniform contraction.

Suppose now that the second condition is fulfilled. For any $d \in \mathcal{G}_{\mathcal{U}'}$ we find that $d \circ (f \times f) \in \mathcal{G}_{\mathcal{U}}$. This proves that if d is a uniformly continuous ∞p-metric on X', $d \circ (f \times f)$ is a uniformly continuous ∞p-metric on X. Hence $f : (X,\mathcal{U}) \to (X',\mathcal{U}')$ is uniformly continuous. □

Corollary 5.1.11 *The functor*

$$\begin{array}{ccc} \textbf{\textit{Unif}} & \to & \textbf{\textit{UG}} \\ (X,\mathcal{U}) & \rightarrowtail & (X, \mathcal{G}_{\mathcal{U}}) \\ f & \rightarrowtail & f \end{array}$$

is a full concrete embedding of ***Unif*** *in* ***UG****.*

As was the case for **TOP** in **AP**, here too we will be able to show that the above embedding is both concretely reflective and concretely coreflective.

Theorem 5.1.12 ***Unif*** *is embedded as concretely reflective subcategory of* ***UG****.*

Proof. It follows at once from the characterization of initial structures in **UG** in theorem 5.1.7 that uniform UG-spaces are closed under the formation of initial structures. □

Proposition 5.1.13 *Given a uniform gauge space $(X, \mathcal{G})$ the collection*

$$\mathcal{U}(\mathcal{G}) := \{U \subset X \times X \mid \exists \epsilon > 0 \exists d \in \mathcal{G} : \{d < \epsilon\} \subset U\}$$

is a uniformity on X. This uniformity is also the uniformity generated by the collection of metrics $\mathcal{G}$.

Proof. First of all we prove that $\mathcal{U}(\mathcal{G})$ is a filter on $X \times X$. It is easy to see that $\mathcal{U}(\mathcal{G})$ is not empty and that the empty set does not belong to $\mathcal{U}(\mathcal{G})$. Suppose that $F \in \mathcal{U}(\mathcal{G})$ and $F \subset G$. Then there exist $\epsilon > 0$ and $d \in \mathcal{G}$ such that $\{d < \epsilon\} \subset F$. Since $F \subset G$, we get that $\{d < \epsilon\} \subset G$ and hence $G \in \mathcal{U}(\mathcal{G})$. Now suppose that $F, G \in \mathcal{U}(\mathcal{G})$. Then there exist $\epsilon_1, \epsilon_2 > 0$ and $d_1, d_2 \in \mathcal{G}$ such that $\{d_1 < \epsilon_1\} \subset F$ and $\{d_2 < \epsilon_2\} \subset G$. Take now $(x, y) \in \{d_1 \vee d_2 < \epsilon_1 \wedge \epsilon_2\}$. This gives us the following implications:

$$\begin{aligned}
&(x, y) \in \{d_1 \vee d_2 < \epsilon_1 \wedge \epsilon_2\} \\
&\Rightarrow d_1 \vee d_2(x, y) < \epsilon_1 \wedge \epsilon_2 \\
&\Rightarrow d_1(x, y) < \epsilon_1 \wedge \epsilon_2 \text{ and } d_2(x, y) < \epsilon_1 \wedge \epsilon_2 \\
&\Rightarrow d_1(x, y) < \epsilon_1 \text{ and } d_2(x, y) < \epsilon_2 \\
&\Rightarrow (x, y) \in \{d_1 < \epsilon_1\} \text{ and } (x, y) \in \{d_2 < \epsilon_2\} \\
&\Rightarrow (x, y) \in F \text{ and } (x, y) \in G \\
&\Rightarrow (x, y) \in F \cap G.
\end{aligned}$$

Hence $\{d_1 \vee d_2 < \epsilon_1 \wedge \epsilon_2\} \subset F \cap G$. This gives us $F \cap G \in \mathcal{U}(\mathcal{G})$, which finishes the first part of the proof.

Secondly we prove that $\mathcal{U}(\mathcal{G})$ is a uniformity on X.
(U1) Suppose that $U \in \mathcal{U}(\mathcal{G})$. Then there exist $\epsilon > 0$ and $d \in \mathcal{G}$ such that $\{d < \epsilon\} \subset U$. Since $\Delta_X \subset \{d < \epsilon\}$, we immediately get $\Delta_X \subset U$.
(U2) Take $U \in \mathcal{U}(\mathcal{G})$. We prove that there exists $V \in \mathcal{G}$ such that $V^2 \subset U$.
We have the existence of $\epsilon > 0$ and a metric $d \in \mathcal{G}$ such that $\{d < \epsilon\} \subset U$. Take $V = \{d < \frac{\epsilon}{2}\}$. It is trivial that $V \in \mathcal{U}(\mathcal{G})$. We now claim that $V^2 \subset U$. This follows from the following implications:

$$\begin{aligned}
&(x, y) \in V^2 \\
&\Rightarrow \exists z \in X \text{ such that } (x, z), (z, y) \in V \\
&\Rightarrow \exists z \in X \text{ such that } d(x, z) < \tfrac{\epsilon}{2} \text{ and } d(z, y) < \tfrac{\epsilon}{2} \\
&\Rightarrow d(x, y) \leq d(x, z) + d(z, y) < \epsilon \\
&\Rightarrow (x, y) \in \{d < \epsilon\} \subset U \\
&\Rightarrow (x, y) \in U.
\end{aligned}$$

(U3) Suppose $U \in \mathcal{U}(\mathcal{G})$. Then we have to prove that $U^{-1} \in \mathcal{U}(\mathcal{G})$. This follows from the fact that all metrics in $\mathcal{G}$ are symmetric.

The theorem also states that $\mathcal{U}(\mathcal{G})$ is the uniformity generated by the collection of metrics $\mathcal{G}$. The uniformity generated by the collection of metrics $\mathcal{G}$ is given by $\sup_{d \in \mathcal{G}} \mathcal{U}_d$, where $\mathcal{U}_d$ is the uniformity generated by d. $\mathcal{U}_d$ has a basis $\{V_\epsilon \mid \epsilon > 0\}$ where $V_\epsilon = \{(x, y) \in X \times X \mid d(x, y) < \epsilon\}$. It is now easy to see that $\sup_{d \in \mathcal{G}} \mathcal{U}_d$ and $\mathcal{U}(\mathcal{G})$ are equal. □

Theorem 5.1.14 ***Unif*** *is embedded as a concretely coreflective subcategory of* ***UG****. For any uniform gauge space* $(X, \mathcal{G})$*, its* ***Unif****-coreflection is given by*

$$\mathrm{id}_X : (X, \mathcal{G}_{\mathcal{U}(\mathcal{G})}) \to (X, \mathcal{G})$$

Proof. That $\mathrm{id}_X : (X, \mathcal{G}_{\mathcal{U}(\mathcal{G})}) \to (X, \mathcal{G})$ is a uniform contraction follows at once from the observation that $\mathcal{G} \subset \mathcal{G}_{\mathcal{U}(\mathcal{G})}$. Suppose now that $(Y, \mathcal{H})$ is a uniform UG-space and that

$$f : (Y, \mathcal{H}) \to (X, \mathcal{G})$$

is a uniform contraction, then it immediately follows that

$$f : (Y, \mathcal{U}(\mathcal{H})) \to (X, \mathcal{U}(\mathcal{G}))$$

is uniformly continuous and hence that

$$f : (Y, \mathcal{H}) \to (X, \mathcal{G}_{\mathcal{U}(\mathcal{G})})$$

is a uniform contraction. □

The next result gives an internal characterization of uniform UG-spaces.

Theorem 5.1.15 *A uniform gauge space $(X, \mathcal{G})$ is a uniform UG-space if and only if $\mathcal{G}$ satisfies the stronger uniform saturation condition which says that if e is a ∞p-metric and*

$$\forall \epsilon > 0 \exists \delta > 0 \exists d \in \mathcal{G} : \{d < \delta\} \subset \{e < \epsilon\}$$

then $e \in \mathcal{G}$.

Proof. This follows immediately from the definitions. □

Given an ∞p-metric space (X, d), we associate with it a natural uniform gauge space in exactly the same way as for **AP**.

Proposition 5.1.16 *Suppose (X, d) is an ∞p-metric space. Then the collection*

$$\mathcal{G}_d := \{e \in pM^{\infty}(X) \mid e \leq d\}$$

is a uniform gauge.

Metric spaces thus have three different forms, as a metric space, as an approach space or as a uniform gauge space. In the last two cases the gauge and the uniform gauge are exactly the same.

Definition 5.1.17 *A uniform gauge space of the type $(X, \mathcal{G}_d)$, for some metric d on X, will be called a* ***metric uniform gauge space*** *or shortly a metric UG-space.*

The fact that $p\mathbf{MET}^{\infty}$ is concretely embedded in **UG** is a consequence of the following proposition.

Proposition 5.1.18 *Let (X, d) and (X', d') be ∞p-metric spaces and consider the function $f : X \to X'$. Then the following are equivalent:*

1. *$f : (X, d) \to (X', d')$ is non-expansive,*
2. *$f : (X, \mathcal{G}_d) \to (X', \mathcal{G}_{d'})$ is a uniform contraction.*

Proof. Suppose that the first condition is fulfilled. Consider an ∞p-metric $e' \leq d'$ on X'. Then we obviously have that $e' \circ (f \times f) \leq d$. Hence $e' \circ (f \times f) \in \mathcal{G}_d$ and we get the second condition.

Now suppose that the second condition is fulfilled. Then we have that $d' \circ (f \times f) \in \mathcal{G}_d$, hence $d' \circ (f \times f) \leq d$ and we get that $f : (X, d) \to (X', d')$ is non-expansive. □

Corollary 5.1.19 *The functor*

$$\begin{array}{ccc} p\boldsymbol{MET}^\infty & \to & \boldsymbol{UG} \\ (X, d) & \rightarrowtail & (X, \mathcal{G}_d) \\ f & \rightarrowtail & f \end{array}$$

is a full concrete embedding of $p\boldsymbol{MET}^\infty$ *in* $\boldsymbol{UG}$.

Given a uniform gauge space $(X, \mathcal{G})$, we define the ∞p-metric

$$d_{\mathcal{G}} := \sup \mathcal{G}.$$

Theorem 5.1.20 $p\boldsymbol{MET}^\infty$ *is embedded as a concretely coreflective subcategory of* $\boldsymbol{UG}$. *For any uniform gauge space* $(X, \mathcal{G})$, *its* $p\boldsymbol{MET}^\infty$*-coreflection is given by*

$$\mathrm{id}_X : (X, \mathcal{G}_{d_{\mathcal{G}}}) \to (X, \mathcal{G}).$$

Proof. That $\mathrm{id}_X : (X, \mathcal{G}_{d_{\mathcal{G}}}) \to (X, \mathcal{G})$ is a uniform contraction, follows from the fact that for any $d \in \mathcal{G}$ we have that $d \leq d_{\mathcal{G}}$. Now suppose that $(Y, d) \in |p\mathbf{MET}^\infty|$ and that

$$f : (Y, \mathcal{G}_d) \to (X, \mathcal{G})$$

is a uniform contraction. Then for all $e \in \mathcal{G}$ we have $e \circ (f \times f) \leq d$ and hence also $d_{\mathcal{G}} \circ (f \times f) \leq d$ which proves that

$$f : (Y, \mathcal{G}_d) \to (X, \mathcal{G}_{d_{\mathcal{G}}})$$

is also a uniform contraction. □

The following theorem gives an internal characterization of metric UG-spaces.

Theorem 5.1.21 *A uniform gauge space* $(X, \mathcal{G})$ *is a metric uniform gauge space if and only if* $\mathcal{G}$ *satisfies any of the following equivalent conditions*

1. $\sup \mathcal{G} \in \mathcal{G}$
2. $\mathcal{G}$ *is closed under the formation of arbitrary suprema.*

Proof. In one direction this follows immediately from the definition of a metric UG-space and in the other direction it suffices to remark that given a uniform gauge $\mathcal{G}$, $\sup \mathcal{G}$ is a metric. □

Just as in the case of **AP**, $p\mathbf{MET}^\infty$ is not embedded reflectively in **UG**. In particular it is not stable under the formation of infinite products. Taking an infinite product of metric uniform gauge spaces in **UG**, provides the underlying set with a uniform gauge structure, which, in general, is neither metric nor uniform but which has as uniform coreflection precisely the product uniformity of the uniformities underlying the metrics. More precisely we have the following theorem.

Theorem 5.1.22 *Any uniform gauge space $(X, \mathcal{G})$ is a subspace of a product of metric spaces in* ***UG****, i.e.* ***UG*** *is the epireflective hull of the subcategory of metric uniform gauge spaces.*

Proof. Let $(X, \mathcal{G})$ be a uniform gauge space then the injective map

$$f : (X, \mathcal{G}) \to \Big(X^{\mathcal{G}}, \prod_{d \in \mathcal{G}} \mathcal{G}_d\Big) : x \mapsto (x)_d$$

is an embedding. We prove this claim. By theorem 5.1.7, we now that a basis for the product uniformity $\prod_{d \in \mathcal{G}} \mathcal{G}_d$ on $X^{\mathcal{G}}$ is given by $\{\sup_{d \in K} d \circ \mathrm{pr}_d \times \mathrm{pr}_d \mid K \in 2^{(\mathcal{G})}\}$. Then a basis for the initial structure on X for the source defined by f is given by

$$\mathcal{H} = \big\{ \sup_{d \in K} d \circ (\mathrm{pr}_d \times \mathrm{pr}_d) \circ (f \times f) \mid K \in 2^{(\mathcal{G})} \big\}.$$

We also have that

$$\begin{aligned} \sup_{d \in K} d \circ (\mathrm{pr}_d \times \mathrm{pr}_d) \circ (f \times f)(x, y) &= \sup_{d \in K} d \circ (\mathrm{pr}_d \times \mathrm{pr}_d)((x)_d, (y)_d) \\ &= \sup_{d \in K} d(x, y), \end{aligned}$$

hence $\mathcal{H} = \{\sup_{d \in K} d \mid K \in 2^{(\mathcal{G})}\}$ and thus $\widehat{\mathcal{H}} = \mathcal{G}$. □

We finish this section by describing the relation between **AP** and **UG**.

Definition 5.1.23 *If $(X, \mathcal{G})$ is a uniform gauge space then the saturation $\tilde{\mathcal{G}}$ is a gauge on X. We call $(X, \tilde{\mathcal{G}})$ the* ***underlying approach space*** *of $(X, \mathcal{G})$.*

If $\mathcal{H} \subset p\mathrm{M}^{\infty}(X)$ is an ideal, then it is at the same time a basis for a uniform gauge and for a gauge. Since the various other associated structures can be derived from a basis for the gauge we immediately obtain the same formulas for the various associated approach structures underlying a uniform gauge space. Thus, if $(X, \mathcal{G})$ is a uniform gauge space with basis $\mathcal{H}$ for $\mathcal{G}$ then we have

$$\delta_{\tilde{\mathcal{G}}}(x, A) = \sup_{d \in \mathcal{H}} \inf_{y \in A} d(x, y)$$

and

$$\lambda_{\tilde{\mathcal{G}}} \mathcal{F}(x) = \sup_{d \in \mathcal{H}} \inf_{F \in \mathcal{F}} \sup_{y \in F} d(x, y)$$

and analogously for the other structures.

Proposition 5.1.24 *Suppose that X and X' are uniform gauge spaces and that $f : X \to X'$ is a function. If f is a uniform contraction, then it is a contraction.*

Proof. This follows immediately from the definitions and the fact that any uniform gauge is a basis for the gauge of the associated approach space. □

Theorem 5.1.25 ***AP*** *is a full subcategory of* ***UG****.*

Proof. This follows from the fact that any gauge is also a uniform gauge and that the definition of uniform contraction is precisely the same as the definition of a contraction. Hence, if $(X, \mathcal{G})$ and $(X', \mathcal{G}')$ are approach spaces and $f : (X, \mathcal{G}) \to (X', \mathcal{G}')$ is a contraction, it is obviously also a uniform contraction when the spaces are considered to be uniform gauge spaces. $\square$

Theorem 5.1.26 *The concrete functor from* ***UG*** *to* ***AP*** *which takes* $(X, \mathcal{G})$ *to* $(X, \tilde{\mathcal{G}})$ *is left adjoint to the embedding of* ***AP*** *into* ***UG****. In other words,* ***AP*** *is coreflectively embedded in* ***UG****.*

Proof. The **AP**-coreflection of a UG-space $(X, \mathcal{G})$ is given by

$$\mathrm{id}_X : (X, \tilde{\mathcal{G}}) \to (X, \mathcal{G}).$$

First of all it is clear that $\mathrm{id}_X : (X, \tilde{\mathcal{G}}) \to (X, \mathcal{G})$ is a uniform contraction.

Consider now $(Y, \mathcal{G}_Y) \in |\mathbf{AP}|$ and a function $f : Y \to X$ such that $f : (Y, \mathcal{G}_Y) \to (X, \mathcal{G})$ is a uniform contraction (when we look at the underlying UG-space of $(Y, \mathcal{G}_Y)$). We need to prove that $f : (Y, \mathcal{G}_Y) \to (X, \tilde{\mathcal{G}})$ is a contraction. Consider therefore $d \in \tilde{\mathcal{G}}$. Then we have that for all $x \in X$, for all $\epsilon > 0$ and for all $\omega < \infty$ there exists $e \in \mathcal{G}$ such that $d(x, \cdot) \wedge \omega \leq e(x, \cdot) + \epsilon$. Since $f : (Y, \mathcal{G}_Y) \to (X, \mathcal{G})$ is a uniform contraction, we get that $e \circ (f \times f) \in \mathcal{G}_Y$. Since we have

$$d \circ (f \times f)(y, \cdot) \wedge \omega \leq e \circ (f \times f)(y, \cdot) + \epsilon,$$

for all $y \in Y$, $\epsilon > 0$ and $\omega < \infty$, we get that $d \circ (f \times f) \in \mathcal{G}_Y$. $\square$

The relations among the categories which we have seen is depicted in the following diagram:

$$\begin{tikzcd}
 & \mathbf{UG} \arrow[r, hook, "c", "r"'] \arrow[dl, hook, "c"'] \arrow[dd] & \mathbf{Unif} \arrow[dd] \\
p\mathbf{MET}^{\infty} \arrow[dr, hook, "c"] & & \\
 & \mathbf{AP} \arrow[r, hook, "c", "r"'] & \mathbf{TOP}
\end{tikzcd}$$

5.2 Function spaces, precompactness and equicontractivity

In order to prove an Ascoli theorem in the setting of approach theory we now require three main items. First of all we need function spaces, secondly we need a notion of precompactness and thirdly we need a notion of equicontractivity. In this section we present a brief study of these three notions.

Suppose we are given two uniform gauge spaces $(X, \mathcal{G}_X)$ and $(Y, \mathcal{G}_Y)$ and let Σ be a cover of X which is closed under finite unions. Let $\mathcal{H}$ be any collection of

functions from X to Y. Then we define a uniform gauge on $\mathcal{H}$ as follows. For any $A \in \Sigma$ and any $d \in \mathcal{G}_Y$ we define

$$D_{d,A} : \mathcal{H} \times \mathcal{H} \to [0, \infty] : (f, g) \mapsto \sup_{x \in A} d\big(f(x), g(x)\big).$$

Clearly, these functions are extended p-metrics and hence they determine a uniform gauge. Actually they form an ideal basis and the uniform gauge generated by this basis is obtained by saturating the set $\{D_{d,A} \mid d \in \mathcal{G}_Y, A \in \Sigma\}$ for the uniform saturation property. We will denote this uniform gauge by $\langle \Sigma, \mathcal{G}_Y \rangle$. The following proposition tells us what is the relation between this structure on $\mathcal{H}$ and well-known function space uniformities and topologies.

Theorem 5.2.1 *If $(X, \mathcal{G}_X)$ and $(Y, \mathcal{G}_Y)$ are uniform gauge spaces, $\mathcal{H} \subset Y^X$ and we consider the uniform gauge $\langle \Sigma, \mathcal{G}_Y \rangle$ on $\mathcal{H}$ then the following hold:*

1. *The topological (respectively uniform) coreflection is structured with the topology (respectively uniformity) of uniform convergence on Σ-sets.*
2. *The extended p-metric coreflection (both in **AP** and in **UG**) is structured with the supremum p-metric.*

Proof. First of all we consider the topological coreflection. Therefore we use theorem 2.2.7, which states that **TOP** is coreflectively embedded in **AP**, and theorem 5.1.26, which states that **AP** is coreflectively embedded in **UG**. Consider now the UG-space $(\mathcal{H}, \langle \Sigma, \mathcal{G}_Y \rangle)$. The **AP**-coreflection is given by $(\mathcal{H}, \langle \Sigma, \mathcal{G}_Y \rangle)$, where we consider $\langle \Sigma, \mathcal{G}_Y \rangle$ as a basis for a gauge instead of a basis for a uniform gauge. The **TOP**-coreflection is then given by the following closure operator. Let B be a subset of $\mathcal{H}$, then the closure of B is defined as follows:

$$\begin{aligned}
\mathrm{cl}(B) &= \{f \in \mathcal{H} \mid \forall D_{d,A} \in \langle \Sigma, \mathcal{G}_Y \rangle \, \exists g \in B : D_{d,A}(f, g) = 0\} \\
&= \big\{f \in \mathcal{H} \mid \forall D_{d,A} \in \langle \Sigma, \mathcal{G}_Y \rangle \, \exists g \in B : \sup_{x \in A} d(f(x), g(x)) = 0\big\} \\
&= \{f \in \mathcal{H} \mid \forall D_{d,A} \in \langle \Sigma, \mathcal{G}_Y \rangle \, \exists g \in B \, \forall x \in A : d(f(x), g(x)) = 0\} \\
&= \{f \in \mathcal{H} \mid \forall d \in \mathcal{G}_Y \, \forall A \in \Sigma \, \exists g \in B \, \forall x \in A : d(f(x), g(x)) = 0\}.
\end{aligned}$$

This is precisely the topology of uniform convergence on Σ-sets.

We now consider the uniform coreflection. The **Unif**-coreflection is given by the uniformity given in proposition 5.1.13. We get the following

$$\begin{aligned}
\mathcal{U}(\langle \Sigma, \mathcal{G}_Y \rangle) &= \{U \subset \mathcal{H} \times \mathcal{H} \mid \exists \epsilon > 0 \, \exists D_{d,A} \in \langle \Sigma, \mathcal{G}_Y \rangle : \{D_{d,A} < \epsilon\} \subset U\} \\
&= \{U \subset \mathcal{H} \times \mathcal{H} \mid \exists \epsilon > 0 \, \exists d \in \mathcal{G}_Y \, \exists A \in \Sigma : \{D_{d,A} < \epsilon\} \subset U\}.
\end{aligned}$$

We also know that $\{D_{d,A} < \epsilon\}$ is the set given by

$$\begin{aligned}
\{D_{d,A} < \epsilon\} &= \big\{(f, g) \in \mathcal{H} \times \mathcal{H} \mid \sup_{x \in A} d(f(x), g(x)) < \epsilon\big\} \\
&= \{(f, g) \in \mathcal{H} \times \mathcal{H} \mid \forall x \in A : d(f(x), g(x)) < \epsilon\} \\
&= \{(f, g) \in \mathcal{H} \times \mathcal{H} \mid d \circ (f_{|A} \times g_{|A}) < \epsilon\}.
\end{aligned}$$

This gives us that the **Unif**-coreflection is precisely the uniformity of uniform convergence on Σ-sets.

To prove the last part, we use theorem 5.1.20, which states that the $p\mathbf{MET}^{\infty}$-coreflection is given by $d_{\langle\Sigma,\mathcal{G}_Y\rangle} = \sup\langle\Sigma,\mathcal{G}_Y\rangle$. This gives us the following, for every $f, g \in \mathcal{H}$:

$$\begin{aligned} d_{\langle\Sigma,\mathcal{G}_Y\rangle}(f,g) &= \sup_{D_{d,A}\in\langle\Sigma,\mathcal{G}_Y\rangle} D_{d,A}(f,g) \\ &= \sup\{D_{d,A}(f,g) \mid d \in \mathcal{G}_Y, A \in \Sigma\} \\ &= \sup\big\{\sup_{x\in A} d(f(x),g(x)) \mid d \in \mathcal{G}_Y, A \in \Sigma\big\} \\ &= \sup\{d(f(x),g(x)) \mid d \in \mathcal{G}_Y, x \in X\}. \end{aligned}$$

We can conclude that $d_{\langle\Sigma,\mathcal{G}_Y\rangle}$ is indeed the supremum p-metric. □

We will now introduce a measure of precompactness on UG-spaces. Before doing so, we repeat the definition of precompactness for uniform spaces.

Definition 5.2.2 *A uniform space $(X,\mathcal{U})$ is called **precompact** if for each $U \in \mathcal{U}$, the cover $\{U(x) : x \in X\}$ of X has a finite subcover.*

Definition 5.2.3 *For a uniform gauge space $(X,\mathcal{G})$ its **measure of precompactness** is defined by*

$$\mu_{pc}(X) = \sup_{d\in\mathcal{G}} \inf_{Y\in 2^{(X)}} \sup_{z\in X} \inf_{x\in Y} d(x,z)$$

It is interesting to compare the formulas for μ_c (for the underlying approach space) and μ_{pc} (for the uniform gauge space itself). What we see is that they are entirely the same except for the first supremum, which in the case of compactness ranges over the set $\mathcal{G}^X$ and in the case of precompactness ranges over the set $\mathcal{G}$. In the approach case, which just as topology is a local theory, the extended p-metrics must be allowed to vary from point to point, and in the uniform gauge case, which just as uniformity is a global theory, the same p-metric has to be chosen in every point.

We can show that for the measure of precompactness a number of good consistency results hold.

Proposition 5.2.4 *1. $\mu_{pc} \leq \mu_c$.*

2. *For a uniform space $(X,\mathcal{U})$ we have that $\mu_{pc}(X) = 0$ if and only if $(X,\mathcal{U})$ is precompact.*

3. *For a metric space (X,d) we have that $\mu_{pc}(X) = 0$ if and only if $(X,\mathcal{U}_d)$ is precompact.*

4. *For a metric space (X,d) we have that $\mu_{pc}(X) < \infty$ if and only if (X,d) is bounded.*

Proof.

1. This follows immediately from the definition of μ_{pc} and the third characterization of μ_c, given in proposition 4.1.3.

2. Suppose first that $(X,\mathcal{U})$ is a uniform space and that $\mu_{pc}(X) = 0$. This means that for all $d \in \mathcal{G}_{\mathcal{U}}$, the collection of all uniformly continuous pseudometrics, we have that $\inf_{Y\in 2^{(X)}} \sup_{z\in X} \inf_{x\in Y} d(x,z) = 0$. This implies that for all $d \in \mathcal{G}_{\mathcal{U}}$, there exists a set $Y \in 2^{(X)}$ such that $\sup_{z\in X} \inf_{x\in Y} d(x,z) = 0$. Therefore we get for every $\epsilon > 0$ and for all $d \in \mathcal{G}_{\mathcal{U}}$ the existence of a set $Y \in 2^{(X)}$ such that $X = \bigcup_{y\in Y} B_d(y,\epsilon)$. Consider now the cover $\{U(x) \mid x \in X\}$ for $U \in \mathcal{U}$. Since $\mathcal{U}$ is generated by the collection of all uniformly continuous pseudometrics, we can find $d \in \mathcal{G}_{\mathcal{U}}$ and $\epsilon > 0$ such that $\{d < \epsilon\} \subset U$. Then we have $\{d(x,\cdot) < \epsilon\} \subset U(x)$, for all $x \in X$. For this pseudometric $d \in \mathcal{G}_{\mathcal{U}}$ and this value of $\epsilon > 0$, we have, due to the foregoing, a set $Y \in 2^{(X)}$ such that $X = \bigcup_{y\in Y} B_d(y,\epsilon)$. For all $y \in Y$, we have that $\{d(y,\cdot) < \epsilon\} \subset U(y)$, hence $\{U(y) \mid y \in Y\}$ is a finite subcover of $\{U(x) \mid x \in X\}$, which is precisely what we needed to prove.
   To prove the other implication, consider a uniform space $(X,\mathcal{U})$ which is precompact. The uniformity $\mathcal{U}$ is generated by the collection of all uniformly continuous pseudometrics. Hence, we have that $\{d < \epsilon\} \in \mathcal{U}$, for all $d \in \mathcal{G}_{\mathcal{U}}$ and every $\epsilon > 0$. Take $\epsilon > 0$ arbitrary. Since $(X,\mathcal{U})$ is precompact, we have that for every $d \in \mathcal{G}_{\mathcal{U}}$ the cover $\{d(x,\cdot) < \epsilon \mid x \in X\}$ has a finite subcover. Hence, for all $d \in \mathcal{G}_{\mathcal{U}}$ there exists $Y \in 2^{(X)}$ such that $X = \bigcup_{y\in Y}\{d(y,\cdot) < \epsilon\}$. This means that for all $d \in \mathcal{G}_{\mathcal{U}}$ there exists $Y \in 2^{(X)}$ such that $\sup_{z\in X} \inf_{x\in Y} d(x,z) < \epsilon$, or, $\sup_{d\in\mathcal{G}_{\mathcal{U}}} \inf_{Y\in 2^{(X)}} \sup_{z\in X} \inf_{x\in Y} d(x,z) < \epsilon$. By arbitrariness of ϵ, this shows that $\mu_{pc}(X) = 0$.

3. This is an immediate consequence of 2., where $\mathcal{U}$ is the uniformity generated by the pseudometric d.

4. For a metric space (X,d) we have $\{d\}$ as a basis for the uniform gauge. Hence, the formula of the measure of precompactness is given by

$$\mu_{pc}(X) = \inf_{Y\in 2^{(X)}} \sup_{z\in X} \inf_{x\in Y} d(x,z).$$

   Hence, for metric spaces, we have that $\mu_c(X) = \mu_{pc}(X)$. This proposition follows immediately from theorem 4.1.6.

□

The following example shows that even in a metric space the measure of precompactness, just as the measure of compactness, can attain any value.

Example 5.2.5

Let $[0,a]$ be equipped with the Euclidean metric and consider the supremum-metric on $X := [0,a]^{\mathbb{N}}$. Then we have

$$\mu_{pc}(X) = \inf_{Y\in 2^{(X)}} \sup_{(z_n)_n\in X} \inf_{(y_n)_n\in Y} \sup_{n\in\mathbb{N}} |y_n - z_n|.$$

This implies that $\mu_{pc}(X) = a$.

Also the inequality $\mu_{pc} \leq \mu_c$ is, in general, strict. To see this, it suffices to take a precompact non-compact uniform space and embed it in **UG**. Then one measure is zero, whereas the other one is infinite.

We will also require the following result.

Proposition 5.2.6 *If X is a uniform gauge space and $A_1, \cdots, A_n$ are subsets of X then $\mu_{pc}\big(\bigcup_{i=1}^n A_i\big) \leq \sup_{i=1}^n \mu_{pc}(A_i)$.*

Proof.

$$\begin{aligned}
\mu_{pc}\left(\bigcup_{i=1}^n A_i\right) &= \sup_{d\in\mathcal{G}} \inf_{Y\in 2^{(\bigcup_{i=1}^n A_i)}} \sup_{z\in\bigcup_{i=1}^n A_i} \inf_{x\in Y} d(x,z) \\
&= \sup_{d\in\mathcal{G}} \inf_{Y\in 2^{(\bigcup_{i=1}^n A_i)}} \sup_{i=1}^n \sup_{z\in A_i} \inf_{x\in Y} d(x,z) \\
&\leq \sup_{i=1}^n \sup_{d\in\mathcal{G}} \inf_{Y\in 2^{(A_i)}} \sup_{z\in A_i} \inf_{x\in Y} d(x,z) \\
&= \sup_{i=1}^n \mu_{pc}(A_i).
\end{aligned}$$

□

Next, we will introduce the concept of equicontractivity. Before doing so, we first give a definition of uniform equicontinuity.

Definition 5.2.7 *A collection $\mathcal{H}$ of maps from a uniform space $(X,\mathcal{U})$ to a uniform space $(X,\mathcal{V})$ is said to be* ***uniformly equicontinuous*** *provided that for each $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $(f \times f)U \subset V$ whenever $f \in \mathcal{H}$.*

Definition 5.2.8 *If $(X,\mathcal{G}_X)$ and $(Y,\mathcal{G}_Y)$ are uniform gauge spaces, then $\mathcal{H} \subset Y^X$ is called* ***uniformly equicontractive*** *if*

$$\forall d \in \mathcal{G}_Y,\ \exists e \in \mathcal{G}_X,\ \forall f \in \mathcal{H} : d \circ (f \times f) \leq e,$$

and it is clear that it is sufficient that the condition is satisfied for all d in a basis of $\mathcal{G}_Y$

Clearly, if $\mathcal{H}$ is uniformly equicontractive then each $f \in \mathcal{H}$ is a uniform contraction, and a subset of a uniformly equicontractive set is again uniformly equicontractive.

Proposition 5.2.9 *In the case of uniform spaces X and Y a set $\mathcal{H} \subset Y^X$ is uniformly equicontractive if and only if it is uniformly equicontinuous, and in the case of p-metric spaces X and Y a set $\mathcal{H} \subset Y^X$ is uniformly equicontractive simply when it consists of uniform contractions, i.e., non-expansive maps.*

Proof. First of all let $(X,\mathcal{U}_X)$ and $(Y,\mathcal{U}_Y)$ be uniform spaces. We know that every uniformity is generated by a family of pseudometrics, so in particular it is generated by the family of all uniformly continuous pseudometrics. Suppose that $\mathcal{H}$ is uniformly equicontractive. We have to prove that for all $V \in \mathcal{U}_Y$ there is $U \in \mathcal{U}_X$ such that $(f \times f)U \subset V$ for all $f \in \mathcal{H}$. It is sufficient to restrict the proof to elements of the basis of $\mathcal{U}_Y$. Hence, we consider $V = \{d < \epsilon\} \in \mathcal{U}_Y$, for a uniformly continuous pseudometric d on $Y \times Y$ and $\epsilon > 0$. We have that there exists $e \in \mathcal{G}_{\mathcal{U}_X}$ such that $d \circ (f \times f) \leq e$. But then we have found a set $U \in \mathcal{U}_X$, namely $U = \{e < \epsilon\}$ such that $(f \times f)U \subset V$, for all $f \in \mathcal{H}$.

To prove the other implication, suppose that $\mathcal{H}$ is uniformly equicontinuous. Take now $d \in \mathcal{G}_{\mathcal{U}_Y}$ and $\epsilon > 0$. Then $\{d < \epsilon\} \in \mathcal{U}_Y$. Because of the fact that $\mathcal{H}$ is uniformly equicontinuous, we can find a set $U \in \mathcal{U}_X$ such that $(f \times f)U \subset \{d < \epsilon\}$, for all $f \in \mathcal{H}$. This is equivalent to saying that $U \subset \{d \circ (f \times f) < \epsilon\}$, for all $f \in \mathcal{H}$. Since $\mathcal{U}_X$ is the uniformity generated by all uniformly continuous pseudometrics on $X \times X$, we can find a uniformly continuous pseudometric $e \in \mathcal{G}_{\mathcal{U}_X}$ such that $\{e < \epsilon\} \subset U$. But then we have $\{e < \epsilon\} \subset \{d \circ (f \times f) < \epsilon\}$. By arbitrariness of ϵ, this shows $d \circ (f \times f) \leq e$, for all $f \in \mathcal{H}$.

Secondly, let (X, d_X) and (Y, d_Y) be metric spaces. Then we have $\{d_X\}$ as a basis for $\mathcal{G}_{d_X}$ and $\{d_Y\}$ as a basis for $\mathcal{G}_{d_Y}$. $\mathcal{H}$ is then uniformly equicontractive if for all $f \in \mathcal{H}$ we have that $d_Y \circ (f \times f) \leq d_X$, in particular if all maps in $\mathcal{H}$ are non-expansive. $\square$

Since in what follows we will explicitly be working with the measure of precompactness, the nicest formulation of an Ascoli theorem is obtained if we also work with natural measures of uniform contractivity and equicontractivity. Taking into account the second characterization of uniform contractions given in the first section, the following measures seem natural.

Definition 5.2.10 *For any $f \in Y^X$ we define the* ***measure of uniform contractivity*** *of f as*

$$\mu_{uc}(f) := \inf\{\delta \mid \forall d \in \mathcal{G}_Y \exists e \in \mathcal{G}_X : d \circ (f \times f) \leq e + \delta\},$$

and obviously then, for any $\mathcal{H} \subset Y^X$ we define the ***measure of uniform equicontractivity*** *of $\mathcal{H}$*

$$\mu_{uec}(\mathcal{H}) := \inf\{\delta \mid \forall d \in \mathcal{G}_Y \exists e \in \mathcal{G}_X \forall f \in \mathcal{H} : d \circ (f \times f) \leq e + \delta\}.$$

There are several good consistency results proving that these are indeed natural concepts in approach theory. In accordance with proposition 5.2.9. for example, it is easily seen that in the case of extended p-metric spaces $\mu_{uec}(\mathcal{H}) = \sup_{f \in \mathcal{H}} \mu_{uc}(f)$. The following example however shows that, in general, this is not the case, and moreover that the measure of uniform equicontractivity can attain all possible values, even for a set where all the individual functions are uniform contractions.

Example 5.2.11

Consider again $X := [0,a]^{\mathbb{N}}$ but now equipped with the product uniform gauge. Then all projections $\mathrm{pr}_n : [0,a]^{\mathbb{N}} \to [0,a]$ are uniform contractions. For $\mathcal{H} := \{\mathrm{pr}_n \mid n \in \mathbb{N}\}$ however, we have $\mu_{uec}(\mathcal{H}) = a$.
Proof. Consider the projections $\mathrm{pr}_n : [0,a]^{\mathbb{N}} \to [0,a] : (x_n)_n \mapsto x_n$. Then we have

$$(d_E \circ pr_n \times pr_n)((x_n)_n, (y_n)_n) = d_E(x_n, y_n) \leq \sup_{n\in\mathbb{N}} d_E(x_n, y_n),$$

hence the functions pr_n are uniform contractions.

We now compute $\mu_{uec}(\mathcal{H})$.

$$\begin{aligned}
\mu_{uec}(\mathcal{H}) &= \inf\left\{\delta \mid \exists e \in \prod \mathcal{G}_{d_E} \forall n \in \mathbb{N} : d_E \circ pr_n \times pr_n \leq e + \delta\right\} \\
&= \inf\{\delta \mid \forall n \in \mathbb{N} : d_E \circ pr_n \times pr_n \leq \delta\} \quad \text{since } d_E \circ pr_n \times pr_n \in \prod \mathcal{G}_{d_E} \\
&= \inf\{\delta \mid \sup_{n\in\mathbb{N}} d_E \circ pr_n \times \mathrm{pr}_n \leq \delta\} \\
&= \inf\{\delta \mid a \leq \delta\} \\
&= a.
\end{aligned}$$

□

5.3 An Ascoli theorem

Ascoli's theorem describes (pre)compact subsets of function spaces. Before giving a formulation and a proof of this theorem, we describe some preliminary results which are interesting in their own right.

Let $(X, \mathcal{G}_X)$ and $(Y, \mathcal{G}_Y)$ be uniform gauge spaces, let Σ be a cover of X and let $\mathcal{H} \subset Y^X$ be an arbitrary collection of maps. Further let $\mathcal{H}$ be equipped with the uniform gauge structure induced by $\langle \Sigma, \mathcal{G}_Y \rangle$. For any $x \in X$ the "point-x-evaluation map" is denoted as follows:

$$\mathrm{ev}_x : \mathcal{H} \to Y : f \mapsto f(x).$$

Further, if $A \subset X$ then we put $\mathcal{H}_{|A} := \{f_{|A} \mid f \in \mathcal{H}\}$.

Proposition 5.3.1 *The measure of precompactness decreases under uniform contractions.*

Proof. Let $(X, \mathcal{G}_X)$ and $(Y, \mathcal{G}_Y)$ be UG-spaces and let $f : X \to Y$ be a uniform contraction. We have to prove that $\mu_{pc}(f(X)) \leq \mu_{pc}(X)$. We have that for any $d \in \mathcal{G}_Y$ $d \circ (f \times f) \in \mathcal{G}_X$. The proof follows from the following computations:

$$\begin{aligned}
\mu_{pc}(f(X)) &= \sup_{d\in\mathcal{G}_Y} \inf_{Y\in 2^{(f(X))}} \sup_{z\in f(X)} \inf_{x\in Y} d(x,z) \\
&= \sup_{d\in\mathcal{G}_Y} \inf_{Y\in 2^{(X)}} \sup_{z\in X} \inf_{x\in Y} d(f(x), f(z)) \\
&\leq \sup_{e\in\mathcal{G}_X} \inf_{Y\in 2^{(X)}} \sup_{z\in X} \inf_{x\in Y} e(x,y) \\
&= \mu_{pc}(X).
\end{aligned}$$

□

The following result implies that if $\mathcal{H}$ is precompact, then so is $\mathrm{ev}_x(\mathcal{H})$, for any $x \in X$.

Proposition 5.3.2 *The following inequality holds:*

$$\sup_{x \in X} \mu_{pc}\big(\mathrm{ev}_x(\mathcal{H})\big) \leq \mu_{pc}(\mathcal{H}).$$

Proof. For any $x \in X$ choose $A \in \Sigma$ such that $x \in A$. Then it follows that $d \circ (f \times f) \leq D_{d,A}$. Hence, for any $x \in X$, the map $\mathrm{ev}_x : \mathcal{H} \to Y$ is a uniform contraction. Hence, the result is an immediate consequence of proposition 5.3.1. □

The following proposition implies that if all functions in $\mathcal{H}$ have uniformly continuous restrictions to A, for a set $A \in \Sigma$, and if $\mathcal{H}$ itself is precompact, then $\mathcal{H}_{|A}$ is uniformly equicontinuous.

Proposition 5.3.3 *For any $A \in \Sigma$ the following inequality holds:*

$$\mu_{uec}(\mathcal{H}_{|A}) \leq 2\mu_{pc}(\mathcal{H}) + \sup_{f \in \mathcal{H}} \mu_{uc}(f_{|A}).$$

Proof. We suppose that all values on the right-hand side are finite, otherwise there is nothing to prove. First of all we choose α such that

$$\mu_{pc}(\mathcal{H}) = \sup_{d \in \mathcal{G}_Y, A \in \Sigma} \inf_{\mathcal{K} \in 2^{(\mathcal{H})}} \sup_{f \in \mathcal{H}} \inf_{g \in \mathcal{K}} \sup_{a \in A} d\big(f(a), g(a)\big) < \alpha. \tag{5.1}$$

Next, we also choose β such that for all $h \in \mathcal{H}$

$$\mu_{uc}(h_{|A}) = \inf\{\delta \mid \forall d \in \mathcal{G}_Y \exists e \in \mathcal{G}_X : d \circ (h_{|A} \times h_{|A}) \leq e + \delta\} < \beta. \tag{5.2}$$

Now fix $d \in \mathcal{G}_Y$ and $A \in \Sigma$. From (5.1) it follows that there exists a finite subset $\mathcal{K} \subset \mathcal{H}$ such that $\sup_{f \in \mathcal{H}} \inf_{g \in \mathcal{K}} \sup_{a \in A} d\big(f(a), g(a)\big) < \alpha$. From (5.2), for any $g \in \mathcal{K}$ there exists $e_g \in \mathcal{G}_X$ such that $d \circ (g_{|A} \times g_{|A}) \leq e_g + \beta$. Put $e := \sup_{g \in \mathcal{K}} e_g \in \mathcal{G}_X$. Then, if $f \in \mathcal{H}$ we can find $g \in \mathcal{K}$ such that for all $a \in A$, $d\big(f(a), g(a)\big) < \alpha$. Hence it follows that for all $x, y \in A$:

$$\begin{aligned} d\big(f(x), f(y)\big) &\leq d\big(f(x), g(x)\big) + d\big(g(x), g(y)\big) + d\big(g(y), f(y)\big) \\ &\leq \alpha + e_g(x, y) + \beta + \alpha \\ &\leq e(x, y) + (2\alpha + \beta). \end{aligned}$$

□

The classical theorem of Ascoli, as we can find in Bourbaki [4], states the following: "*If each set in Σ is precompact, if for each set $A \in \Sigma$ the collection $\mathcal{H}_{|A}$ is uniformly equicontinuous and if for each $x \in X$, $\mathrm{ev}_x(\mathcal{H})$ is precompact, then $\mathcal{H}$ is precompact*". This, finally, is a consequence of the following Ascoli theorem, which, analogously to the foregoing propositions, again has no conditions, since they are "encapsulated in the inequality".

Theorem 5.3.4 *The following inequality holds:*

$$\frac{1}{2}\mu_{pc}(\mathcal{H}) \leq \sup_{x\in X} \mu_{pc}\big(\operatorname{ev}_x(\mathcal{H})\big) + \sup_{A\in\Sigma} \mu_{pc}(A) + \sup_{A\in\Sigma} \mu_{uec}(\mathcal{H}_{|A}).$$

Proof. Again, we suppose that all values on the right-hand side are finite. Choose α and β such that, for all $x \in X$ and $A \in \Sigma$ we have that

$$\mu_{pc}\big(\operatorname{ev}_x(\mathcal{H})\big) = \sup_{x\in X} \sup_{d\in\mathcal{G}_Y} \inf_{\mathcal{F}\in 2^{(\mathcal{H})}} \sup_{f\in\mathcal{H}} \inf_{g\in\mathcal{F}} d\big(f(x), g(x)\big) < \alpha, \tag{5.3}$$

$$\mu_{pc}(A) = \sup_{A\in\Sigma} \sup_{e\in\mathcal{G}_X} \inf_{B\in 2^{(A)}} \sup_{a\in A} \inf_{b\in B} e(a,b) < \beta. \tag{5.4}$$

Next we also choose γ such that for all $A \in \Sigma, \mu_{uec}(\mathcal{H}_{|A}) < \gamma$, which implies that

$$\forall d \in \mathcal{G}_Y \exists e \in \mathcal{G}_X \forall k \in \mathcal{H}_{|A} : d \circ (k \times k) \leq e + \gamma. \tag{5.5}$$

Now fix $d \in \mathcal{G}_Y$ and $A \in \Sigma$. From (5.5) it follows that there exists $e \in \mathcal{G}_X$ such that for all $f \in \mathcal{H}$:

$$d \circ (f_{|A} \times f_{|A}) \leq e + \gamma.$$

For this e, from (5.4) it then follows that there exists a finite subset $B \subset A$ and a function $A \to B : a \mapsto b_a$ such that $e(a, b_a) < \beta$.

Let $Z := \bigcup_{b\in B} \operatorname{ev}_b(\mathcal{H}) \subset Y$. Then it follows from (5.3) and proposition 5.2.6 that

$$\mu_{pc}(Z) = \mu_{pc}\left(\bigcup_{b\in B} \operatorname{ev}_b(\mathcal{H})\right) \leq \sup_{b\in B} \mu_{pc}\big(\operatorname{ev}_b(\mathcal{H})\big) < \alpha.$$

Hence, there exists a finite subset $C \subset Z$ and a function $Z \to C : z \mapsto c_z$ such that $d(z, c_z) < \alpha$. For an $h \in C^B$ let $\mathcal{B}(h) := \{f \in \mathcal{H} \mid \forall b \in B : d(f(b), h(b)) < \alpha\}$.

Now, fix $f \in \mathcal{H}$, and consider the function $h_f : B \to C : b \mapsto c_{f(b)}$. It then follows that $f \in \mathcal{B}(h_f)$. Let $\mathcal{K} := \{h \in C^B \mid \mathcal{B}(h) \neq \emptyset\}$. Then the foregoing shows that the collection $\{\mathcal{B}(h) \mid h \in \mathcal{K}\}$ is a finite cover of $\mathcal{H}$. Now for each $h \in \mathcal{K}$ we choose an arbitrary function $g_h \in \mathcal{B}(h)$ and we let $\mathcal{F} := \{g_h \mid h \in \mathcal{K}\}$. Then $\mathcal{F}$ is a finite subset of $\mathcal{H}$ and, by the foregoing, we obtain that for any $a \in A$

$$\begin{aligned}
d\big(f(a), g_{h_f}(a)\big) &\leq d\big(f(a), f(b_a)\big) + d\big(f(b_a), h_f(b_a)\big) \\
&\quad + d\big(h_f(b_a), g_{h_f}(b_a)\big) + d\big(g_{h_f}(b_a), g_{h_f}(a)\big) \\
&= d\big(f(a), f(b_a)\big) + d\big(f(b_a), c_{f(b_a)}\big) \\
&\quad + d\big(h_f(b_a), g_{h_f}(b_a)\big) + d\big(g_{h_f}(b_a), g_{h_f}(a)\big) \\
&\leq \big(e(a,b_a) + \gamma\big) + \alpha + \alpha + \big(e(a,b_a) + \gamma\big) \\
&= 2\big(e(a,b_a) + \alpha + \gamma\big) \\
&\leq 2(\alpha + \beta + \gamma),
\end{aligned}$$

which by the arbitrariness of respectively $a \in A, d \in \mathcal{G}_Y$ and $A \in \Sigma$ shows that $\mu_{pc}(\mathcal{H}_{|A}) \leq 2(\alpha + \beta + \gamma)$.

□

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